Tutorial on Categorical Models for (Negation-Free) Linear Logic

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Crash Course: Linear Logic

Linear Logic: Brief History

- In 1986, Jean-Yves Girard studied the semantics for intuitionistic logic, in particular he developed *coherence spaces* as semantics for system F (second-order typed λ-calculus) [20].
- Girard made a key observation about coherence spaces: the function space of stable maps $A \Rightarrow B$ between two coherent spaces A and B, modeling *intuitionistic implication*, decomposed naturally into

$$A \Rightarrow B = !A \multimap B$$

  where $!A$ is a space of repetitions and $!A \multimap B$ is the function space of *linear maps*.
- This prompted Girard to develop a new logic with *linear implication* and the *exponential connective* ! at its core:

Linear logic

  → *Substructural*: controlled introduction of contraction and weakening via exponential connectives
  → *Resource-sensitive*: logic of resources, not stable truths
  → *Refinement of classical and intuitionistic logic*: symmetry of classical logic and constructivism of intuitionistic logic

Classical Logic: Gentzen's System LK

To motivate linear logic, we will start from structural rule considerations.

- In 1934-1935, Gentzen introduced an innovation to proof theory and mathematical logic: the *sequent calculus* and his system LK (LJ) [1].

→ Formulas in system LK are given by the following grammar

$$A ::= a \mid \neg A \mid A \wedge A \mid 1 \mid A \vee A \mid 0$$

- $\neg$ is **negation**, $\wedge$ is **conjunction** and $\vee$ is **disjunction**.
- The **implication** connective $\Rightarrow$ is defined by

$$A \Rightarrow B := \neg A \vee B$$

→ A **sequent** is an expression

$$\Gamma \vdash \Delta, \quad \text{where} \quad \Gamma = A_1, \ldots A_n \quad \text{and} \quad \Delta = B_1, \ldots B_m$$

are finite sequences of formulae, which represents

$$A_1 \text{ and} \ldots \text{and } A_n \quad \text{implies} \quad B_1 \text{ or } \ldots \text{ or } B_m$$

→ A **formal proof** is a *derivation tree* constructed according to the rules of LK, which are separated into three classes: **axiom**, **structural**, and **logical** rules.

Classical Logic: Gentzen's System LK

Axiom Rule

$$\frac{}{A \vdash A}\ (\text{ax})$$

Structural Rules

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\text{cut}) \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (\text{exch}_l) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\text{exch}_r)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\text{weak}_l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (\text{weak}_r) \qquad \frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\text{cont}_l) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (\text{cont}_r)$$

Logical Rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_{l_1}) \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_{l_2}) \qquad \frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2 \vdash B, \Delta_2}{\Gamma_1, \Gamma_2 \vdash A \wedge B, \Delta_1, \Delta_2}\ (\wedge_r)$$

$$\frac{\Gamma_1, A \vdash \Delta_1 \quad \Gamma_2, B \vdash \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \vdash \Delta_1, \Delta_2}\ (\vee_l) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee_{r_1}) \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ (\vee_{l_2})$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, 1 \vdash \Delta}\ (1_l) \qquad \frac{}{\vdash 1}\ (1_r) \qquad \frac{}{0 \vdash}\ (0_l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash 0, \Delta}\ (0_r)$$

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ (\neg_l) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ (\neg_r)$$

- Negation can be defined solely on atomic formula and then extended by recursively applying **de Morgan duality**:

$$\neg(a) := \neg a \qquad \neg(\neg a) := a$$
$$\neg 1 := 0 \qquad \neg 0 := 1$$
$$\neg(A \wedge B) := (\neg A) \vee (\neg B) \qquad \neg(A \vee B) := (\neg A) \wedge (\neg B)$$

- Note the perfect *symmetry* of system LK, with negation allowing us to pass from the right hand side of $\vdash$ to the left!

$$\frac{A \vdash B}{\vdash \neg A, B}\ (\neg_r) \qquad \frac{\vdash \neg A, B}{A \vdash B}\ (\neg_l)$$

$\rightarrow$ Classical logic can be formulated as a *one-sided* (right-sided) sequent calculus with only 4 rules.

- We can alternatively have implication $\Rightarrow$ as a primitive built in connective instead of negation and defining negation by

$$\neg A := A \Rightarrow 0$$

Structural Rules: Deep dive

→ Structural rules manipulate the formulas of a sequent but do not alter them.

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\text{cut})$$

- Modus Ponens deductive principle: *If A implies B and A holds, then B holds.*
- Composition of proofs.

$$\frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (\text{exch}_l) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\text{exch}_r)$$

- Order of hypotheses and conclusions is unimportant.
- Commutativity on conjunction and disjunction.

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\text{weak}_l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash A, \Delta}\ (\text{weak}_r)$$

- $A \Rightarrow B$ may be established by proving B, without using the hypothesis A.
- Causes without effects or fake dependencies.

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ (\text{cont}_l) \qquad \frac{\Gamma \vdash A, A, \Delta}{\Gamma \vdash A, \Delta}\ (\text{cont}_r)$$

- $A \Rightarrow B$ is proved when B is deduced from A, possibly used many times.
- Formulas represent stable truths.

Cut-Elimination and Intuitionistic Logic

- **Cut-elimination procedure**: evaluates a proof π by converting it into a proof π' eliminating occurrences of cut-rule while preserving the conclusion.
- LK satisfies the **cut-elimination theorem**: every proof can be transformed into a cut-free proof.
  - $\rightarrow$ Cut-elimination procedure terminates.
- Cut-elimination procedure in LK is highly *non-deterministic*. A proof can be reduced to two different cut-free proofs.
  - $\rightarrow$ Classical logic cannot be given non-trivial semantics: all proofs of a sequent will have the same denotation.
- How can cut-elimination be made a *confluent*?

$\rightarrow$ One solution: restrict the sequent calculus LK to **intuitionistic sequents**:

$$\Gamma \vdash A$$

sequents with exactly one formula on the right-hand side.

- This system was designated LJ by Gentzen and is a formalization of **intuitionistic logic**.
- Intuitionistic logic is then *constructive*:
  - $\rightarrow$ Correspondence between proofs and programs/algorithms:

Curry-Howard isomorphism

- Confluence is obtained in LJ by limiting the weakening and contraction rules to only be applicable to the left-hand side of sequents.
- While LJ is constructive, it loses the symmetry of LK.
- Alternative approach: What if we keep the symmetry, but limit the structural rules of contraction and weakening instead?

Sub-structural logics

- Notable examples of a sub-structural logic studied prior to linear logic:
  - Relevance logics: permitting contraction, but not weakening [2]
  - BCK-logics: logics without contraction [28]
  - Lambek's syntactic calculus: weakening, contraction and exchange are absent [26]

→ *Linear logic* (LL) took a novel approach: restricting contraction and weakening to sequents explicitly marked by additional modalities ! and ?.

  - ! for weakening and contraction on the left
  - ? for weakening and contraction on the right

- Key consequence of absence of general contraction and weakening:
  $\rightarrow$ split of conjunction and disjunction into **multiplicative** and **additive**
- In LK, there are two equivalent versions of introduction rules for conjunction and disjunction,e.g. for conjunction:

Context-free (multiplicative)

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_l^m) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Theta \vdash B, \Psi}{\Gamma, \Theta \vdash A \wedge B, \Delta, \Psi}\ (\wedge_r^m)$$

Context-dependent (additive)

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_{l_1}^a) \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_{l_2}^a) \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ (\wedge_r^a)$$

- These are equivalent due to contraction and weakening:

$$\dfrac{\dfrac{\Gamma, A \vdash \Delta}{\Gamma, A, B \vdash \Delta}\ (\text{weak}_l)}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_l^m) \qquad \dfrac{\dfrac{\dfrac{\Gamma, B \vdash \Delta}{\Gamma, B, A \vdash \Delta}\ (\text{weak}_l)}{\Gamma, A, B \vdash \Delta}\ (\text{exch}_l)}{\Gamma, A \wedge B \vdash \Delta}\ (\wedge_l^m) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A, B \vdash \Delta}{\Gamma, A, A \wedge B \vdash \Delta}\ (\wedge_{l_2}^a)}{\Gamma, A \wedge B, A \vdash \Delta}\ (\text{exch}_l)}{\Gamma, A \wedge B, A \wedge B \vdash \Delta}\ (\wedge_{l_1}^a)}{\Gamma, A \wedge B \vdash \Delta}\ (\text{cont}_l)$$

- In LL, there are two conjunction and two disjunction connectives, one with multiplicative introduction rules and one with additive rules.

Linear Logic: Girard's LL

→ Formulas are given by the following grammar

$$A ::= a \mid A^\perp \mid A \otimes A \mid \mathbf{1} \mid A \parr A \mid \perp \mid A \& A \mid \top \mid A \oplus A \mid \mathbf{0} \mid\ !A \mid\ ?A$$

- Multiplicatives:
  - $(-)^\perp$ is **linear negation**, read **perp**
  - $\otimes$ is multiplicative conjunction, read **tensor**, with constant **1** as multiplicative truth
  - $\parr$ is multiplicative disjunction, read **par**, with constant $\perp$ as multiplicative falsity
- Additives:
  - & is additive conjunction, read **with**, with constant $\top$ as additive truth
  - $\oplus$ is additive disjunction, read **plus**, with constant **0** as additive falsity
- Exponentials:
  - ! is the exponential modality known as **of course** or **bang**
  - ? is the exponential modality known as **why not** or **whimper**
- The binary connective of **linear implication** $\multimap$ is defined by:

$$A \multimap B := A^\perp \parr B$$

Axiom Rule

$$\frac{}{A \vdash A}\ (\mathrm{ax})$$

Structural Rules

$$\frac{\Gamma_1 \vdash A, \Delta_1 \quad \Gamma_2, A \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (\mathrm{cut}) \qquad \frac{\Gamma_1, A, B, \Gamma_2 \vdash \Delta}{\Gamma_1, B, A, \Gamma_2 \vdash \Delta}\ (\mathrm{exch}_l) \qquad \frac{\Gamma \vdash \Delta_1, A, B, \Delta_2}{\Gamma \vdash \Delta_1, B, A, \Delta_2}\ (\mathrm{exch}_r)$$

Multiplicative Rules

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^\perp \vdash \Delta}\ ((-)_l^\perp) \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash A^\perp, \Delta}\ ((-)_r^\perp)$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \otimes B \vdash \Delta}\ (\otimes_l) \qquad \frac{\Gamma \vdash A, \Delta \quad \Theta \vdash B, \Psi}{\Gamma, \Theta \vdash A \otimes B, \Delta, \Psi}\ (\otimes_r)$$

$$\frac{\Gamma \vdash A, \Delta \quad \Theta, B \vdash \Psi}{\Gamma, \Theta, A \parr B \vdash \Delta, \Psi}\ (\parr_l) \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \parr B, \Delta}\ (\parr_r)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, \mathbf{1} \vdash \Delta}\ (\mathbf{1}_l) \qquad \frac{}{\vdash \mathbf{1}}\ (\mathbf{1}_r) \qquad \frac{}{\perp \vdash}\ (\perp_l) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \perp, \Delta}\ (\perp_r)$$

Additive Rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ (\&_{l_1}) \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \& B \vdash \Delta}\ (\&_{l_2}) \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \& B, \Delta}\ (\&_r)$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma A \oplus B \vdash \Delta}\ (\oplus_l) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \oplus B \Delta}\ (\oplus_{r_1}) \qquad \frac{\Gamma \vdash B, \Delta}{\Gamma \vdash A \oplus B \Delta}\ (\oplus_{l_2})$$

$$\frac{}{\Gamma \vdash \top, \Delta}\ (\top_r) \qquad \frac{}{\Gamma, \mathbf{0} \vdash \Delta}\ (\mathbf{0}_l)$$

Exponential Rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!_l) \qquad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ (!_r) \qquad \frac{!\Gamma, A \vdash ?\Delta}{!\Gamma, ?A \vdash ?\Delta}\ (?_l) \qquad \frac{\Gamma \vdash A, \Delta}{\Gamma \vdash ?A, \Delta}\ (?_r)$$

$$\frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\text{weak}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash ?A, \Delta}\ (?\text{weak}) \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\text{cont}) \qquad \frac{\Gamma \vdash ?A, ?A, \Delta}{\Gamma \vdash ?A, \Delta}\ (?\text{cont})$$

Note: Introduction rules for ! and ? often grouped as:

- Rules $(!_l)$ and $(?_r)$ are known as **dereliction** rules (!der) and (?der)
- Rules $(!_r)$ and $(?_l)$ are known as **promotion** rules (!prom) and (?prom)

Linear Logic: Key Facts

- Linear negation $(-)^\perp$ can be defined by **de Morgan duality**:

$$
\begin{aligned}
(a)^\perp &:= a^\perp & (a^\perp)^\perp &:= a \\
\mathbf{1}^\perp &:= \bot & \bot^\perp &:= \mathbf{1} \\
(A \otimes B)^\perp &:= A^\perp \parr B^\perp & (A \parr B)^\perp &:= A^\perp \otimes B^\perp \\
\top^\perp &:= \mathbf{0} & \mathbf{0}^\perp &:= \top^\perp \\
(A \mathbin{\&} B)^\perp &:= A^\perp \oplus B^\perp & (A \oplus B)^\perp &:= A^\perp \mathbin{\&} B^\perp \\
(!A)^\perp &:= ?A^\perp & (?A)^\perp &:= !A^\perp
\end{aligned}
$$

- We can have linear implication $\multimap$ as a *primitive* instead of negation:

$$A^\perp := A \multimap \bot$$

- Key propositional equivalences:

$$!(A \mathbin{\&} B) \equiv !A \otimes !B \quad ?(A \oplus B) \equiv ?A \parr ?B \quad !\top \equiv \mathbf{1} \quad ?\mathbf{0} \equiv \bot$$

Exponentiation converts addition to multiplication

$$A \otimes (B \oplus C) \equiv (A \otimes B) \oplus (A \otimes C) \quad A \parr (B \mathbin{\&} C) \equiv (A \parr B) \mathbin{\&} (A \parr C)$$

$$A \otimes \mathbf{0} \equiv \mathbf{0} \equiv \mathbf{0} \otimes A \qquad A \parr \top \equiv \top \equiv \top \parr A$$

Multiplication distributes over addition

- Without contraction and weakening, hypothesis must be relevant to the implication and cannot be reused.
  - $\rightarrow$ Formulas are resources which are consumed through entailment.
  - $\rightarrow$ In the sequent $A \vdash B$, A is consumed to obtain B.
  - $\rightarrow$ Exponential modalities allow the controlled use of stable truths.

Linear logic: logic of resources

- Linear logic is perfectly symmetrical like classical logic.
  - $\rightarrow$ Two sided-sequents with *linear negation* passing from one side to the other.
  - $\rightarrow$ Linear logic can thus be formulated as a one-sided (right-sided) sequent calculus with fewer rules.
- Linear logic is constructive like intuitionistic logic.
  - $\rightarrow$ Well-behaved cut-elimination for LL.
  - $\rightarrow$ Cut-elimination is confluent up to *rule commutation* (order in which rules are applied) [22], yielding non-trivial denotational semantics

Linear logic: refinement of classical logic and intuitionistic logic

Linear Logic: Fragments and Variants

- Various fragments of linear logic (subsets of the connectives with their respective rules) are studied:

	MLL^+	MLL	$MALL^+$	MALL	$MELL^+$	MELL	LL^+	LL
$\otimes, \mathbf{1}, \parr, \bot$	✓	✓	✓	✓	✓	✓	✓	✓
$(-)^\bot, \multimap$		✓		✓		✓		✓
$\&, \top, \oplus, \mathbf{0}$			✓	✓			✓	✓
$!, ?$					✓	✓	✓	✓

- Many variants to LL were introduced, most importantly

Intuitionistic linear logic

  - → Obtained by restricted to intuitionistic sequents, as with LK versus LJ.
  - → LL as introduced previously is often known now as *classical* linear logic.
- In this talk, we shall be exclusively concerned with classical linear logic.

Linear Logic: Girard Translation

- Girard Translation $(\quad)^o$
  $\rightarrow$ Embeds intuitionistic logic into linear logic
  For formulas, the Girard translation is inductively defined by:

$$a^o = a \qquad 1^o = 1 \qquad 0^o = 0$$

$$(A \wedge B)^o = A^o \,\&\, B^o \qquad (A \vee B)^o = !A^o \oplus !B^o$$

$$(A \Rightarrow B)^o = !A^o \multimap B \qquad (\neg A)^o = !A^o \multimap \mathbf{0}$$

  For proofs, the Girard translation translates a proof of a sequent

$$A_1, \ldots A_n \vdash B$$

  into a proof of a sequent

$$!A_1^o, \ldots !A_n^o \vdash B^o$$

  inductively (based on the length of the deduction).
- Faithful: A^o is provable in LL if and only if A is provable in LJ.

- *Proof theory*
  - $\rightarrow$ Study of formal proofs as mathematical structures in their own right
- Fundamental idea of categorical semantics:

Formulas	$\leftrightarrow$	Objects
Proofs	$\leftrightarrow$	Morphisms

- Consider a logic $\mathcal{L}$. Let us define a category of denotations $\mathbb{X}$.
  - For every formula A, we associate an object $[\![A]\!]$ in the category $\mathbb{X}$.
  - For every proof π, we associate a morphism $[\![\pi]\!]$ in $\mathbb{X}$, known as its **denotations** or **interpretation**,

$$\dfrac{\overset{\pi}{\vdots}}{\Gamma \vdash \Delta} \qquad \leftrightarrow \qquad [\![\pi]\!] : [\![\Gamma]\!] \rightarrow [\![\Delta]\!]$$

  such that the denotation is *invariant* under cut-elimination:

$$\pi \rightsquigarrow \pi' \rightarrow [\![\pi]\!] = [\![\pi']\!]$$

  - Now recall that $\Gamma = A_1, \ldots A_n$ and $\Delta = B_1, \ldots B_m$.
    - $\rightarrow$ $[\![\Gamma]\!]$ should depend on $[\![A_i]\!]$ and $[\![\Delta]\!]$ should depend on $[\![B_j]\!]$.

Categorical Semantics: Brief Intro

- Why a category? Suppose there is the *cut-rule* and *axiom rule* in a logic $\mathcal{L}$. Given proofs π and π', we can obtain a new proof π'' by the cut-rule.

$$\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A \vdash B} \qquad \dfrac{\begin{array}{c}\pi'\\ \vdots\end{array}}{B \vdash C} \quad \to \quad \dfrac{\dfrac{\begin{array}{c}\pi\\ \vdots\end{array}}{A \vdash B} \qquad \dfrac{\begin{array}{c}\pi'\\ \vdots\end{array}}{B \vdash C}}{A \vdash C}$$

  Denotation $[\![\pi'']\!]$ should be built from $[\![\pi]\!]$ and $[\![\pi']\!]$.

- There needs to be a binary operation

$$[\![A]\!] \xrightarrow{[\![\pi]\!]} [\![B]\!] \xrightarrow{[\![\pi']\!]} [\![C]\!] \qquad \mapsto \qquad [\![A]\!] \xrightarrow{[\![\pi'']\!]} [\![C]\!]$$

- This is precisely categorical composition!

$$[\![\pi'']\!] = [\![\pi]\!];[\![\pi']\!]$$

  $\to$ Invariance under cut-elimination implies it must be associative.

- Axiom rule gives the identity morphisms:

$$\dfrac{}{A \vdash A}\,(\text{ax}) \qquad \leftrightarrow \qquad 1_{[\![A]\!]} : [\![A]\!] \to [\![A]\!]$$

  $\to$ Invariance under cut-elimination implies it must be unital.

Categorical Semantics: Brief Intro

- Consider now the linear logic fragment MLL^+.
  - There are binary connectives $\otimes$ and $⅋$.
  - $[\![A \otimes B]\!]$ and $[\![A ⅋ B]\!]$ should be defined by $[\![A]\!]$ and $[\![B]\!]$.

  $\rightarrow$ $\mathbb{X}$ should support such binary operations on objects, which are denoted by the same symbols $\otimes$ and $⅋$:

$$[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!] \qquad [\![A ⅋ B]\!] = [\![A]\!] ⅋ [\![B]\!]$$

  - Given proof π, then by $\otimes_l$ and $⅋_r$,

$$\frac{\begin{array}{c}\pi\\ \vdots\end{array}}{A_1, \ldots A_n \vdash B_1, \ldots B_m} \quad \rightarrow \quad \frac{\begin{array}{c}\pi'\\ \vdots\end{array}}{A_1 \otimes \ldots \otimes A_n \vdash B_1 ⅋ \cdots ⅋ B_m}$$

$$\leftrightarrow \qquad [\![\pi]\!] : [\![A_1]\!] \otimes \ldots \otimes [\![A_n]\!] \rightarrow [\![B_1]\!] ⅋ \cdots ⅋ [\![B_m]\!]$$

  - The logical rules tell us what additional conditions $\otimes$ and $⅋$ must satisfy:

    $\rightarrow$ $\otimes$ and $⅋$ are symmetric monoidal products, with units $[\![\mathbf{1}]\!]$ and $[\![\bot]\!]$
  - Given a fragment of LL, we say $\mathbb{X}$ is a model for the fragment if
    - each formula and proof has a denotation in $\mathbb{X}$, and
    - the denotations are invariant under cut-elimination.

Material from this section adapted from [12, 18, 21, 27, 30]

Categorical Models of MLL and MLL^+

Definition (Barr [3])

A $*$**-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, *)$ is a symmetric monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$, with an adjoint equivalence $(-)^* \dashv (-)^{*op} : \mathbb{X} \to \mathbb{X}^{op}$ such that there is a natural isomorphism

$$\mathbb{X}(A \otimes B, C^*) \cong \mathbb{X}(A, (B \otimes C)^*)$$

$\Rightarrow$ Categorical models for MLL (as shown by Seely [29]):

$*$**-autonomous categories**

- MLL: de Morgan dualities relate $\otimes/\mathbf{1}$ and $⅋/\bot$

Proposition (Folklore)

Given a $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$, define the following par structure by

$$A ⅋ B = (A^* \otimes B^*)^* \qquad \bot = \mathbf{1}^*$$

then $(\mathbb{X}, ⅋, \bot)$ is a symmetric monoidal category and we have isomorphisms

$$\mathbb{X}(A \otimes B, C) \cong \mathbb{X}(A, B^* ⅋ C)$$

- MLL: linear implication is defined by $A \multimap B = A^\perp \parr B$

Proposition (Barr [3])

A $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ *is closed, with internal hom defined by*

$$A \multimap B = (A \otimes B^*)^*$$

- MLL: linear implication can be made primitive and linear negation defined by

$$A^\perp = A \multimap \perp$$

Definition (Barr [4])

A **$*$-autonomous category** $(\mathbb{X}, \otimes, \mathbf{1}, \multimap, \perp)$ consists of:

- a symmetric monoidal closed category $(\mathbb{X}, \otimes, \mathbf{1}, \multimap)$ with
- a **dualizing object** $\perp$: an object $\perp$ such that the canonical map

$$d_A : A \to (A \multimap \perp) \multimap \perp$$

is an isomorphism for all A.

Note: d_A is obtained by currying the evaluation map $ev_{A,\perp} : A \otimes (A \multimap \perp) \to \perp$.

Example

1. **Sup**: category of complete (sup-)lattices and sup-preserving maps [3]
   - *Tensor product*: $M \otimes N$ is the standard tensor product of sup-lattices, classifying maps $M \times N \to P$ preserving sups in both variables
   - *Linear negation*: P^* is the opposite poset P^{op}, while $f^* : N^* \to M^*$ obtained from the inf-preserving right adjoint $f : M \to N$
   - *Linear implication*: $M \multimap N$ is the complete lattice of sup-preserving maps $f : M \to N$, ordered pointwise
   - *Dualizing object*: $\perp$ is the opposite poset Ω^{op} of truth-values

2. Compact closed categories: symmetric monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ where each A has a dual A^* with maps $\eta_A : \mathbf{1} \to A^* \otimes A$ and $\epsilon_A : A \otimes A^* \to \mathbf{1}$.
   - *Linear negation*: A^* is the dual of A, with $f^* : B^* \to A^*$ defined using the unit and counit maps
   - *Linear implication*: $A \multimap B$ is given by $(A \otimes B^*)^*$
   - *Dualizing object*: $\perp$ is the monoidal unit $\mathbf{1}$

Proposition (Dold, Puppe [19])

A $*$-autonomous category satisfying $(A \otimes B)^* \cong A^* \otimes B^*$ is compact closed.

$*$-Autonomous Categories: Examples

Example

3. **Rel**: category of sets and relations $\rightarrow$ compact closed category
   - *Tensor product*: $X \otimes Y$ is cartesian products of sets $X \times Y$
   - *Linear negation*: X^* is simply X, while $R^* : Y \rightarrow X$ is the transpose of $R : X \rightarrow Y$
   - *Linear implication*: $X \multimap Y$ is the cartesian product of sets $X \times Y$
   - *Dualizing object*: $\perp$ is the singleton set $\{\star\}$

4. **Chu construction** [13]

   Consider a symmetric closed monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ with pullbacks and an object $\perp \in \mathbb{X}$, let $\mathrm{Chu}(\mathbb{X}, \perp)$ denote the category of:
   - objects: triples (A, A', α), where A, A' are objects and $\alpha : A \otimes A' \rightarrow \perp$ is a map in $\mathbb{X}$,
   - morphisms: $(f, g) : (A, A', \alpha) \rightarrow (B, B', \beta)$ are pairs of maps $f : A \rightarrow B$ and $g : B' \rightarrow A'$ such that

$$
\begin{array}{ccc}
A \otimes B' & \xrightarrow{f \otimes 1_{B'}} & B \otimes B' \\
{\scriptstyle 1_A \otimes g} \downarrow & & \downarrow {\scriptstyle \beta} \\
A \otimes A' & \xrightarrow[\alpha]{} & \perp
\end{array}
$$

Example

4. **Chu construction** [13]
   - *Tensor product*: $(A, A', \alpha) \otimes (B, B', \beta)$ is given by $((A \otimes B) \otimes P, \nu)$ where

$$\begin{array}{ccc} P & \longrightarrow & B \multimap A' \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle B \multimap \alpha^t} \\ A \multimap B' & \xrightarrow[A \multimap \beta^t]{} & (A \otimes B) \multimap \bot \end{array}$$

and ν is the transpose of $P \to (A \otimes B) \multimap \bot$

   - *Linear negation*: $(A, B, \alpha)^*$ is given by $(B, A, B \otimes A \xrightarrow{\sigma_\otimes} A \otimes B \xrightarrow{\alpha} \bot)$
   - *Dualizing object*: $\bot$ is the triple $(\mathbf{1}, \bot, \mathbf{1} \otimes \bot \xrightarrow{u^L_\otimes} \bot)$

From $*$-Autonomous to Linearly Distributive

- Categorical models for MLL provided by $*$-autonomous categories make multiplicative conjunction and linear negation/linear implication primitive.
- Multiplicative disjunction is then extra structure.

Can we consider alternative semantics which put conjunction and disjunction on a "level playing field"?

- Cockett and Seely introduced alternative semantics in 1992: **linearly distributive categories** [14].
- *Motivation*:
  - $\rightarrow$ take multiplicative conjunction $\otimes$ and disjunction $\parr$ as primitive
  - $\rightarrow$ connect to presentation of LL as a two-sided sequent calculus
  - $\rightarrow$ provide categorical models for MLL^+
  - $\rightarrow$ promote the importance of the distributivity between $\otimes$ and $\parr$

Definition (Cockett, Seely [14])

A **linearly distributive category**, or **LDC**, $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ consists of:

- a category $(\mathbb{X}, ;, 1_A)$,
- a **tensor** monoidal structure $(\mathbb{X}, \otimes, \mathbf{1})$,
- a **par** monoidal structure $(\mathbb{X}, \parr, \bot)$, and
- left and right **linear distributivity** natural transformations

$$\delta^R_{A,B,C} \colon (A \parr B) \otimes C \to A \parr (B \otimes C)$$

$$\delta^L_{A,B,C} \colon A \otimes (B \parr C) \to (A \otimes B) \parr C$$

satisfying coherence conditions.

- Linear distributivities are precisely what is needed to model the cut rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,(\text{ax}) \qquad \dfrac{}{C \vdash C}\,(\text{ax})}{B \parr C \vdash B, C}\,(\parr_l) \qquad \dfrac{\dfrac{}{A \vdash A}\,(\text{ax}) \qquad \dfrac{}{B \vdash B}\,(\text{ax})}{A, B \vdash A \otimes B}\,(\otimes_r)}{A, B \parr C \vdash A \otimes B, C}\,(\text{cut})}{A \otimes (B \parr C) \vdash A \otimes B, C}\,(\otimes_l)}{A \otimes (B \parr C) \vdash (A \otimes B) \parr C}\,(\parr_r)$$

Definition (Cockett, Seely [14])

A LDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ is **symmetric**, or a **SLDC**, if the $\otimes$ and $\parr$ monoidal structures are symmetric with braidings $\sigma_\otimes$ and $\sigma_\parr$ respectively, and

$$\begin{array}{ccc} (A \parr B) \otimes C & \xrightarrow{\delta^R_{A,B,C}} & A \parr (B \otimes C) \\ {\sigma_\otimes}_{A \parr B, C} \downarrow & & \uparrow {\sigma_\parr}_{B \otimes C, A} \\ C \otimes (A \parr B) & & (B \otimes C) \parr A \\ 1_C \otimes {\sigma_\parr}_{A,B} \downarrow & & \uparrow {\sigma_\otimes}_{C,B} \parr 1_A \\ C \otimes (B \parr A) & \xrightarrow[\delta^L_{C,B,A}]{} & (C \otimes B) \parr A \end{array}$$

$\Rightarrow$ Categorical models for MLL^+ (as shown by Cockett and Seely [14]):

symmetric linearly distributive categories

Remark. Notational conflict

	Tensor	Par	With	Plus
Girard	$\otimes, \mathbf{1}$	$\parr, \bot$	$\&, \top$	$\oplus, \mathbf{0}$
Cockett & Seely	$\otimes, \top$	$\oplus, \bot$	$\times, 1$	$+, 0$

Example (Cockett, Seely [14])

1. Every $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ is a SLDC, with par given by de Morgan duality
$$A \parr B = (A^* \otimes B^*)^* \qquad \bot = \mathbf{1}^*$$
and linear distributivities defined as follows: first define
$$\dfrac{\dfrac{B^* \otimes A^* \xrightarrow{\sigma_\otimes} A^* \otimes B^* \cong (A^* \otimes B^*)^{**}}{B^* \to (A^* \otimes (A^* \otimes B^*)^*)^*}}{A^* \otimes (A^* \otimes B^*)^* \to B}$$
then define the right linear distributivity by
$$\dfrac{\dfrac{\dfrac{(A^* \otimes (A^* \otimes B^*)^*) \otimes C \to B \otimes C \cong (B \otimes C)^{**}}{((A^* \otimes B^*)^* \otimes C) \otimes A^* \to (B \otimes C)^{**}}}{(A^* \otimes B^*)^* \otimes C \to (A^* \otimes (B \otimes C)^*)^*}}{(A \parr B) \otimes C \xrightarrow{\delta^R} A \parr (B \otimes C)}$$
and similar for the left.

Example (Cockett, Seely [14])

2. Every SMC $(\mathbb{X}, \otimes, \mathbf{1})$ can be viewed as a **degenerate** SLDC, with

$$\otimes = \parr \qquad \mathbf{1} = \bot \qquad \delta^R_{A,B,C} = \alpha_{\otimes A,B,C} \qquad \delta^L_{A,B,C} = \alpha_{\otimes}{}^{-1}_{A,B,C}$$

3. Consider a symmetric monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ with an object $\bot$, which has a $\otimes$-inverse $\bot^{-1}$, meaning there are maps

$$s^L : \bot \otimes \bot^{-1} \to \mathbf{1} \qquad s^R : \bot^{-1} \otimes \bot \to \mathbf{1}$$

satisfying a coherence condition and define the **shifted tensor**

$$A \parr B = A \otimes (\bot^{-1} \otimes B)$$

Then, $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ is a SLDC with invertible linear distributivities.

Proposition (Cockett, Seely [14])

For every LDC with invertible linear distributivities, the $\parr$-unit $\bot$ has a $\otimes$-inverse, $\bot^{-1} := \mathbf{1} \parr \mathbf{1}$, and moreover $\parr$ is naturally equivalent to the $\bot$-shifted tensor.

Example (Cockett, Seely [14])

4. Consider a SLDC $(\mathbb{X}, \otimes, \mathbf{1}, ⅋, \bot)$ and a comm. $\otimes$-bialgebra

$$\langle B, \nabla_B : B \otimes B \to B, u_B : \mathbf{1} \to B, \Delta_B : B \to B \otimes B, e_B : B \to \mathbf{1} \rangle$$

Define the category $\mathrm{Mod}(B)$:

- objects: left modules of B $\langle M, \alpha_M : B \otimes M \to M \rangle$,
- maps: $f : \langle M, \alpha_M \rangle \to \langle N, \alpha_N \rangle$ is a map $f : M \to N$ such that $\alpha_M; f = (1_B \otimes f); \alpha_N$.

$\mathrm{Mod}(B)$ is a SLDC, with

$$\langle M, \alpha_M \rangle \otimes \langle N, \alpha_N \rangle = \langle M \otimes N, \alpha_{M \otimes N} \rangle \qquad \langle M, \alpha_M \rangle ⅋ \langle N, \alpha_N \rangle = \langle M ⅋ N, \alpha_{M ⅋ N} \rangle$$

$$\alpha_{M \otimes N} : B \otimes (M \otimes N) \xrightarrow{\Delta_B \otimes 1} (B \otimes B) \otimes (M \otimes N) \cong (B \otimes M) \otimes (B \otimes N) \xrightarrow{\alpha_M \otimes \alpha_N} M \otimes N$$

$$\alpha_{M ⅋ N} : B \otimes (M ⅋ N) \xrightarrow{\Delta_B \otimes 1} (B \otimes B) \otimes (M ⅋ N) \xrightarrow{\delta'} (B \otimes M) ⅋ (B \otimes N) \xrightarrow{\alpha_M ⅋ \alpha_N} M ⅋ N$$

and linear distributivities inherited from $\mathbb{X}$.

Remark

$\mathrm{Mod}(B)$ *is* $*$*-autonomous iff* B *is a Hopf algebra.*

SLDCs: Adding negation

Definition (Cockett, Seely [14])

A SLDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ has **negation** if there is an object function $(-)^\bot$, together with the following parametrized family of maps

$$\gamma_A^R : A \otimes A^\bot \to \bot \qquad\qquad \tau_A^R : \mathbf{1} \to A \parr A^\bot$$

which additionally induce the following families

$$\gamma_A^L = A^\bot \otimes A \xrightarrow{\sigma_\otimes} A \otimes A^\bot \xrightarrow{\gamma_A^R} \bot \qquad\qquad \tau_A^L = \mathbf{1} \xrightarrow{\tau_A^R} A \parr A^\bot \xrightarrow{\sigma_\parr} A^\bot \parr A \quad \text{such that}$$

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ {\scriptstyle u^R_{\otimes A}}\downarrow & & \uparrow{\scriptstyle u^L_{\parr A}} \\ A \otimes \mathbf{1} & & \bot \parr A \\ {\scriptstyle 1_A \otimes \tau_A^L}\downarrow & & \uparrow{\scriptstyle \gamma_A^R \parr 1_A} \\ A \otimes (A^\bot \parr A) & \xrightarrow[\delta^L_{A,A^\bot,A}]{} & (A \otimes A^\bot) \parr A \end{array} \qquad \begin{array}{ccc} A^\bot & \xrightarrow{1_{A^\bot}} & A^\bot \\ {\scriptstyle u^R_{\otimes A^\bot}}\downarrow & & \uparrow{\scriptstyle u^L_{\parr A^\bot}} \\ A^\bot \otimes \mathbf{1} & & \bot \parr A^\bot \\ {\scriptstyle 1_{A^\bot} \otimes \tau_A^R}\downarrow & & \uparrow{\scriptstyle \gamma_A^L \parr 1_{A^\bot}} \\ A^\bot \otimes (A \parr A^\bot) & \xrightarrow[\delta^L_{A^\bot,A,A^\bot}]{} & (A^\bot \otimes A) \parr A^\bot \end{array}$$

→ If a SLDC has negation, then $(A, A^\bot, \gamma_A^R, \tau_A^L)$ and $(A^\bot, A, \gamma_A^L, \tau_A^R)$ form **complementation pairs** [15] or **linear duals** [16].

SLDCs with Negation: Model for MLL

Lemma (Cockett, Seely [14])

Consider a SLDC with negation, then there are adjunctions

$$A \otimes (-) \dashv A^\perp \parr (-) \qquad A^\perp \otimes (-) \dashv A \parr (-)$$

corresponding to the following bijections

$$\frac{A \otimes B \to C}{B \to A^\perp \parr C} \qquad \frac{A^\perp \otimes B \to C}{B \to A \parr C}$$

- Above is sufficient to define $(-)^\perp$ on maps and all the necessary isomorphism (*involutive* nature of $(-)^\perp$ and the *de Morgan dualities*) to recover $*$-autonomy:

Theorem (Cockett, Seely [14])

The notions of SLDCs with negation and $*$-autonomous categories coincide.

$\Rightarrow$ Categorical models for MLL (as shown by Cockett and Seely [14]):

$*$-autonomous categories $\Leftrightarrow$ SLDCs with negation

Grishin SLDCs: Model for MLL

- MLL: $\multimap$ can be made primitive instead of $(-)^\perp$

Definition (Cockett, Seely [17])

A SLDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \perp)$ is **Grishin** if

- its $\otimes$-monoidal structure is closed $(\mathbb{X}, \otimes, \mathbf{1}, \multimap)$, and
- the following canonical map is an isomorphism

$$(A \multimap B) \parr C \to A \multimap (B \parr C) \quad \text{for } A, B, C \in \mathbb{X}$$

- *Note:* The above canonical map is obtained by currying the following

$$A \otimes ((A \multimap B) \parr C) \xrightarrow{\delta^L} (A \otimes (A \multimap B)) \parr C \xrightarrow{ev \parr 1} B \parr C$$

- The following equivalence is the so-called **Grishin law** [25] and holds in MLL:

$$(A \multimap B) \parr C \equiv A \multimap (B \parr C)$$

Theorem (Cockett, Seely [14])

The notions of SLDCs with negation and Grishin SLDCs coincide.

- Setting $A^\perp = A \multimap \perp$, γ_A^R is given by closure of $\otimes$, but the Grishin isomorphism guarantees we can coherently form the maps $\tau_A^R : \mathbf{1} \to A \parr A^\perp$.

Categorical Models of MALL and MALL^{+}

- Multiplicative conjunction is modeled by a monoidal product, but additive conjunction is modeled by a binary product.

$\Rightarrow$ Categorical models for MALL (as shown by Seely [29]):

$*$-**autonomous categories** with **finite products**

- *Warning*: Seely used the term "linear categories" to refer to models of MALL. This is no longer in use. In fact, linear categories are most commonly used for models of (intuitionistic) MELL.
- MALL: de Morgan dualities relate $\&/\top$ and $\oplus/\mathbf{0}$
- MALL: multiplicatives distribute over the additives

Lemma

Given $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ with finite products $(\&, \top)$, define the following plus structure by

$$A \oplus B = (A^* \,\&\, B^*)^* \qquad \mathbf{0} = \top^*$$

then $(\oplus, \mathbf{0})$ are finite coproducts in $\mathbb{X}$. Moreover, the following canonical maps are isomorphisms

$$A \parr \top \to \top \qquad \mathbf{0} \to A \otimes \mathbf{0}$$

$$A \parr (B \,\&\, C) \to (A \parr B) \,\&\, (A \parr C) \qquad (A \otimes B) \oplus (A \otimes C) \to A \otimes (B \oplus C)$$

Example

1. **Sup**: category of complete (sup-)lattices and sup-preserving maps
   - *Binary products*: $M \,\&\, N$ is the cartesian product of sets $M \times N$, with pointwise ordering
   - *Terminal object*: $\top$ is the truth-value lattice Ω
2. **Rel**: category of sets and relations $\rightarrow$ compact closed category
   - *Binary products*: $X \,\&\, Y$ is the disjoint union of sets $X \sqcup Y$
   - *Terminal object*: $\top$ is the empty set $\emptyset$
3. **Chu construction** [23]
   Consider a symmetric monoidal closed category $(\mathbb{X}, \otimes, \mathbf{1})$ with pullbacks, *finite products* and *coproducts*, and an object $\bot \in \mathbb{X}$, then $\mathrm{Chu}(\mathbb{X}, \bot)$ has finite products.
   - *Binary products*: $(A, A', \alpha) \,\&\, (B, B', \beta)$ is $(A \,\&\, B, A' \oplus B', \alpha \,\&\, \beta)$ where

$$\alpha \,\&\, \beta : (A \,\&\, B) \otimes (A' \oplus B)' \cong ((A \,\&\, B) \otimes A') \oplus ((A \,\&\, B) \otimes B') \xrightarrow{[(\pi^0 \otimes 1);\alpha, (\pi^1 \otimes 1);\beta]} \bot$$

   - *Terminal object*: $(\top, \mathbf{0}, \top \otimes \mathbf{0} \cong \mathbf{0} \xrightarrow{b_\bot} \bot)$

- From the perspective of MALL^+ and models based on SLDCs, there is no linear negation and de Morgan dualities.
  $\rightarrow$ $\&/\top$ and $\oplus/\mathbf{0}$ must both be taken as primitive.
- MALL^+: the multiplicatives distribute over the additives.
- This was developed by Cockett and Seely in 1999:

Definition (Cockett, Seely [15])

A SLDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ has a **linear terminal object** $(\top, \mathbf{0})$ if it has a terminal object $\top$ and an initial object $\mathbf{0}$ such that the canonical maps are isomorphisms for all A:

$$\mathbf{0} \xrightarrow{b_{A\otimes\mathbf{0}}} A \otimes \mathbf{0} \qquad A \parr \top \xrightarrow{t_{A\parr\top}} \top$$

A SLDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ has a **linear binary products** $(\&, \oplus)$ if it has binary products $\&$ and coproducts $\oplus$ such that the canonical maps are isomorphisms for all A, B, C:

$$A \parr (B \& C) \xrightarrow{\langle 1\parr p^0, 1\parr p^1\rangle} (A \parr B) \& (A \parr C) \qquad (A \otimes B) \oplus (A \otimes C) \xrightarrow{[1\otimes\iota^0, 1\otimes\iota^1]} A \otimes (B \oplus C)$$

⇒ Categorical models for MALL^{+} (as shown by Cockett and Seely [15]):

SLDCs with **finite linear products**

Example

1. Every $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ with finite products $(\&, \top)$ is a SLDC with finite linear products, with plus structure given by de Morgan duality [15]:

$$A \oplus B = (A^{*} \,\&\, B^{*})^{*} \qquad \mathbf{0} = \top^{*}$$

2. **More examples?**

⇒ Categorical models for MALL (as shown by Cockett and Seely [15]):

$*$**-autonomous categories** with **finite products**

$\Leftrightarrow$

SLDCs with **negation** and **finite linear products**

Tangent: Linear Functors and Transformations

Monoidal Functors and Transformations

- Cockett and Seely formulated the appropriate definition of functor and transformation between LDCs in 1999 [15].
- The idea was to obtain an inclusion 2-functor $U : *\text{-sAUT} \to \text{SLDC}$.

Lemma

Consider a (symmetric) lax monoidal functor $(F, m^0, m^2) : (\mathbb{X}, \otimes, \mathbf{1}, *) \to (\mathbb{Y}, \otimes, \mathbf{1}, *)$ *between* $*$*-autonomous categories. Define the following*

$$\bar{F} : \mathbb{X} \to \mathbb{X} \qquad A \mapsto (F(A^*))^*$$

$$n^2_{A,B} : \bar{F}(A \parr B) \cong (F(A^* \otimes B^*))^* \xrightarrow{(m^2_{A^*,B^*})^*} (F(A^*) \otimes F(B^*))^* \cong \bar{F}(A) \parr \bar{F}(B)$$

$$n^0 : \bar{F}(\bot) \cong (F(\mathbf{1}))^* \xrightarrow{(m^0)^*} \mathbf{1}^* = \bot$$

Then, $(\bar{F}, n^2, n^0) : (\mathbb{X}, \parr, \bot) \to (\mathbb{Y}, \parr, \bot)$ *is a colax monoidal functor.*

Consider now a monoidal transformation $\alpha : (F, m^0, m^2) \Rightarrow (G, m^2, m^0)$*. Define*

$$\bar{\alpha}_A : \bar{G}(A) = (G(A^*))^* \xrightarrow{(\alpha_{A^*})^*} (F(A^*))^* = \bar{F}(A)$$

Then, $\bar{\alpha} : (\bar{G}, n^2, n^0) \Rightarrow (\bar{F}, n^2, n^0)$ *is a monoidal transformation.*

Definition (Cockett, Seely [15])

A **linear functor** $F = (F_\otimes, F_\parr) : \mathbb{X} \to \mathbb{Y}$ between LDCs $\mathbb{X}$ and $\mathbb{Y}$ consists of:

- a lax $\otimes$-monoidal functor $(F_\otimes, m_{\mathbf{1}}, m_\otimes)\colon (\mathbb{X}, \otimes, \mathbf{1}) \to (\mathbb{Y}, \otimes, \mathbf{1})$,
- a colax $\parr$-monoidal functor $(F_\parr, n_\perp, n_\parr)\colon (\mathbb{X}, \parr, \perp) \to (\mathbb{Y}, \parr, \perp)$, and
- **linear strengths** natural transformations,

$$v^R_{\otimes A,B}\colon F_\otimes(A \parr B) \to F_\parr(A) \parr F_\otimes(B) \qquad v^L_{\otimes A,B}\colon F_\otimes(A \parr B) \to F_\otimes(A) \parr F_\parr(B)$$
$$v^R_{\parr A,B}\colon F_\otimes(A) \otimes F_\parr(B) \to F_\parr(A \otimes B) \qquad v^L_{\parr A,B}\colon F_\parr(A) \otimes F_\otimes(B) \to F_\parr(A \otimes B)$$

subject to various coherence conditions.

If $\mathbb{X}$ and $\mathbb{Y}$ are SLDCs, the linear functor is **symmetric** if $F_\otimes$ and $F_\parr$ are symmetric monoidal functors, and the following conditions hold:

$$v^R_\otimes; \sigma_\parr = F_\otimes(\sigma_\parr); v^L_\otimes \qquad v^R_\parr; F_\parr(\sigma_\otimes) = \sigma_\otimes; v^L_\parr$$

As such, one can omit either the left or right linear strengths from the definition.

Consider a (symmetric) lax monoidal functor $(F, m^0, m^2) : (\mathbb{X}, \otimes, \mathbf{1}, *) \to (\mathbb{Y}, \otimes, \mathbf{1}, *)$, then $v^R_{\otimes A,B} : F(A \parr B) \to \bar{F}(A) \parr F(B)$ is given by:

$$\dfrac{\dfrac{\dfrac{(A^* \otimes B^*)^* \xrightarrow{=} (A^* \otimes B^*)^*}{(A^* \otimes B^*)^* \otimes A^* \to B^{**}}}{(A^* \otimes B^*)^* \otimes A^* \to B^{**} \cong B}}{F((A^* \otimes B^*)^* \otimes A^*) \to F(B)}$$

$$\dfrac{\dfrac{F((A^* \otimes B^*)^*) \otimes F(A^*) \xrightarrow{m^2} F((A^* \otimes B^*)^* \otimes A^*) \to F(B)}{F((A^* \otimes B^*)^*) \otimes F(A^*) \to F(B) \cong F(B)^{**}}}{F((A^* \otimes B^*)^*) \to (F(A^*) \otimes F(B)^*)^* \cong ((F(A^*))^{**} \otimes F(B)^*)^*}$$

Alternatively, if the $*$-autonomous categories are viewed as SLDCs with negation, then $v^R_{\otimes A,B} : F(A \parr B) \to \bar{F}(A) \parr F(B)$ is given by:

$$F(A \parr B) \cong \mathbf{1} \otimes F(A \parr B) \xrightarrow{\tau \otimes 1} ((F(A^\perp))^\perp \parr F(A^\perp)) \otimes F(A \parr B) \xrightarrow{\delta^R}$$

$$(F(A^\perp))^\perp \parr (F(A^\perp) \otimes F(A \parr B)) \xrightarrow{1 \parr m^2} (F(A^\perp))^\perp \parr F(A^\perp \otimes (A \parr B)) \xrightarrow{1 \parr F(\delta^R)}$$

$$(F(A^\perp))^\perp \parr F((A^\perp \otimes A) \parr B) \xrightarrow{1 \parr F(\gamma \parr 1)} (F(A^\perp))^\perp \parr F(\perp \parr B) \cong (F(A^\perp))^\perp \parr F(B)$$

Definition (Cockett, Seely [15])

A **linear transformation** $\alpha = (\alpha_\otimes, \alpha_\parr) : F \Rightarrow G : \mathbb{X} \to \mathbb{Y}$ consists of:

- a $\otimes$-monoidal transformation ${\alpha_\otimes}_A : F_\otimes(A) \Rightarrow G_\otimes(A)$, and
- a $\parr$-monoidal transformation ${\alpha_\parr}_A : G_\parr(A) \Rightarrow F_\parr(A)$,

which commute coherently with the linear strengths of F and G:

$$
\begin{array}{ccc}
F_\otimes(A \parr B) & \xrightarrow{{\alpha_\otimes}_{A \parr B}} & G_\otimes(A \parr B) \\
 & & \downarrow {v^R_\otimes}^G_{A,B} \\
{v^R_\otimes}^F_{A,B} \downarrow & & G_\parr(A) \parr G_\otimes(B) \\
 & & \downarrow {\alpha_\parr}_A \parr 1_{G_\otimes(B)} \\
F_\parr(A) \parr F_\otimes(B) & \xrightarrow[1_{F_\parr(A)} \parr {\alpha_\otimes}_B]{} & F_\parr(A) \parr G_\otimes(B)
\end{array}
$$

$$\alpha_\otimes ; {v^R_\otimes}^G ; (\alpha_\parr \parr 1) = {v^R_\otimes}^F ; (1 \parr \alpha_\otimes)$$
$$\alpha_\otimes ; {v^L_\otimes}^G ; (1 \parr \alpha_\parr) = {v^L_\otimes}^F ; (\alpha_\otimes \parr 1)$$
$$(\alpha_\otimes \otimes 1) ; {v^R_\parr}^G ; \alpha_\parr = (1 \otimes \alpha_\parr) ; {v^R_\parr}^F$$
$$(1 \otimes \alpha_\otimes) ; {v^L_\parr}^G ; \alpha_\parr = (\alpha_\parr \otimes 1) ; {v^L_\parr}^F$$

In the symmetric context, we can drop the left or right versions of the coherence conditions above.

Theorem (Cockett, Seely [15])

There is an inclusion of 2-categories $U : *\text{-}sAUT \to SLDC$.

Recall that a category having a finite (co)products may be formulated in terms of *functors and transformations*:

- A category $\mathbb{X}$ has a terminal object $\top$ iff there is a constant functor $\top : \mathbb{1} \to \mathbb{X}$ and a natural transformation

$$t : 1_{\mathbb{X}} \Rightarrow \top' : \mathbb{X} \to \mathbb{X} \qquad t_A : A \to \top$$

  such that

$$t_\top = \mathrm{id}_\top : \top \Rightarrow \top : \mathbb{1} \to \mathbb{X} \quad \Leftrightarrow \quad t_\top = 1_\top$$

- A category $\mathbb{X}$ has binary products $A \,\&\, B$ iff there is a functor $\& : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ and natural transformations

$$\Delta : 1_{\mathbb{X}} \Rightarrow \mathrm{copy}_{\mathbb{X}}; \& : \mathbb{X} \to \mathbb{X} \qquad \pi^i : \& \Rightarrow \mathrm{proj}^i_{\mathbb{X},\mathbb{X}} : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$$

$$\Delta_A : A \to A \,\&\, A \qquad \pi^i_{A_0,A_1} : A_0 \,\&\, A_1 \to A_i$$

  such that

$$\Delta; \pi^i_{\mathrm{copy}_{\mathbb{X}}} = \mathrm{id}_{1_{\mathbb{X}}} : 1_{\mathbb{X}} \Rightarrow 1_{\mathbb{X}} : \mathbb{X} \to \mathbb{X} \quad \Leftrightarrow \quad \Delta_A; \pi^i_{A,A} = 1_A$$

$$\Delta_\&; (\pi^0 \,\&\, \pi^1) = \mathrm{id}_\& : \& \Rightarrow \& : \mathbb{X} \times \mathbb{X} \to \mathbb{X} \quad \Leftrightarrow \quad \Delta_{A\&B}; (\pi^0_{A,B} \,\&\, \pi^1_{A,B}) = 1_{A\&B}$$

→ *Dually* for initial objects and binary coproducts.

→ If $\mathbb{X}$ is a SMC, then the above functors and transformations are (trivially) symmetric monoidal in the appropriate sense.

Proposition (Cockett, Seely [15])

Given a SLDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$*, there is a linear terminal object* $(\top, \mathbf{0})$ *iff there is a constant linear functor* $(\top, \mathbf{0}) : \mathbb{1} \to \mathbb{X}$ *and a linear transformation*

$$(t, b) : (1_{\mathbb{X}}, 1_{\mathbb{X}}) \Rightarrow (\top', \mathbf{0}') \qquad t_A : A \to \top, \qquad b_A : \mathbf{0} \to A$$

such that $(t_\top, b_\mathbf{0}) = (\mathrm{id}_\top, \mathrm{id}_\mathbf{0}) : (\top, \mathbf{0}) \Rightarrow (\top, \mathbf{0}) \quad \Leftrightarrow \quad t_\top = 1_\top$ *and* $b_\mathbf{0} = 1_\mathbf{0}$*.*

Moreover, there are linear binary products $(\&, \oplus)$ *iff there is a linear functor* $(\&, \oplus) : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ *and linear transformations*

$$(\Delta, \nabla) : (1_{\mathbb{X}}, 1_{\mathbb{X}}) \Rightarrow (\mathrm{copy}_{\mathbb{X}}, \mathrm{copy}_{\mathbb{X}}); (\&, \oplus) \qquad \Delta_A : A \to A \,\&\, A, \quad \nabla_A : A \oplus A \to A$$

$$(\pi^i, \iota^i) : (\&, \oplus) \Rightarrow (\mathrm{proj}^i_{\mathbb{X},\mathbb{X}}, \mathrm{inj}^i_{\mathbb{X},\mathbb{X}}) \qquad \pi^i_{A_0,A_1} : A_0 \,\&\, A_1 \to A_i, \quad \iota^i_{A_0,A_1} : A_i \to A_0 \oplus A_1$$

such that

$$(\Delta, \nabla); (\pi^i_{\mathrm{copy}_{\mathbb{X}}}, \iota^i_{\mathrm{copy}_{\mathbb{X}}}) = (\mathrm{id}_{1_{\mathbb{X}}}, \mathrm{id}_{1_{\mathbb{X}}}) \quad \Leftrightarrow \quad \Delta_A; \pi^i_{A,A} = 1_A \text{ and } \iota^i_{A,A}; \nabla_A = 1_A$$

$$(\Delta_\&, \nabla_\oplus); (\pi^0 \,\&\, \pi^1, \iota^0 \oplus \iota^1) = (\mathrm{id}_\&, \mathrm{id}_\oplus) \quad \Leftrightarrow$$
$$\Delta_{A\&B}; (\pi^0_{A,B} \,\&\, \pi^1_{A,B}) = 1_{A\&B} \text{ and } (\iota^0_{A,B} \oplus \iota^1_{A,B}); \nabla_{A\oplus B} = 1_{A\oplus B}$$

Sketch.

($\Rightarrow$) Suppose $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$ has linear binary products $(\&, \oplus)$, we need to define appropriate linear strengths:

$v^R_{\otimes(A,B),(C,D)} : (A \parr C) \& (B \parr D) \to (A \oplus B) \parr (C \& D)$ is given by

$$(A \parr C) \& (B \parr D) \xrightarrow{(\iota^0 \parr 1) \& (\iota^1 \parr 1)} ((A \oplus B) \parr C) \& ((A \oplus B) \parr D) \cong (A \oplus B) \parr (C \& D)$$

$v^R_{\parr(A,B),(C,D)} : (A \& B) \otimes (C \oplus D) \to (A \otimes C) \oplus (B \otimes D)$ is given by

$$(A \& B) \otimes (C \oplus D) \cong ((A \& B) \otimes C) \oplus ((A \& B) \otimes D) \xrightarrow{(\pi^0 \otimes 1) \oplus (\pi^1 \otimes 1)} (A \otimes C) \oplus (B \otimes D)$$

($\Leftarrow$) Suppose there are the relevant linear functors and transformations on $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \bot)$, it remains to show distributivity:

$\langle 1 \parr p^0, 1 \parr p^1 \rangle : A \parr (B \& C) \to (A \parr B) \& (A \parr C)$ has inverse given by

$$(A \parr B) \& (A \parr C) \xrightarrow{v^R_\otimes} (A \oplus A) \parr (B \& C) \xrightarrow{\nabla \parr 1} A \parr (B \& C)$$

$[1 \otimes \iota^0, 1 \otimes \iota^1] : (A \otimes B) \oplus (A \otimes C) \to A \otimes (B \oplus C)$ has inverse given by

$$A \otimes (B \oplus C) \xrightarrow{\Delta \otimes 1} (A \& A) \otimes (B \oplus C) \xrightarrow{v^R_\parr} (A \otimes B) \oplus (A \otimes C)$$

Categorical Models of MELL and MELL^+

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\mathrm{der}) \qquad \frac{!\Gamma \vdash A, ?\Delta}{!\Gamma \vdash !A, ?\Delta}\ (!\mathrm{prom}) \qquad \frac{\Gamma \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\mathrm{weak}) \qquad \frac{\Gamma, !A, !A \vdash \Delta}{\Gamma, !A \vdash \Delta}\ (!\mathrm{cont})$$

- The category of denotations should have the following structure:

$$\frac{\overset{\pi}{\vdots}}{A \vdash B} \mapsto \dfrac{\dfrac{\overset{\pi}{\vdots}}{A \vdash B}\ (!\mathrm{der})}{\dfrac{!A \vdash B}{!A \vdash !B}\ (!\mathrm{prom})} \quad \leftrightarrow \quad [\![\pi]\!] : [\![A]\!] \to [\![B]\!] \mapsto ![\![\pi]\!] : [\![!A]\!] \to [\![!B]\!]$$

$$\dfrac{\dfrac{}{A \vdash A}\ (\mathrm{ax})}{!A \vdash A}\ (!\mathrm{der}) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\ (\mathrm{ax})}{!A \vdash A}\ (!\mathrm{der})}{!A \vdash !A}\ (!\mathrm{prom})}{!A \vdash !!A}\ (!\mathrm{prom}) \quad \leftrightarrow \quad \begin{matrix} p^!_A : [\![!A]\!] \to [\![!!A]\!] \\ d^!_A : [\![!A]\!] \to [\![A]\!] \end{matrix}$$

$$\dfrac{\dfrac{}{\vdash \mathbf{1}}\ (\mathbf{1}_r)}{!A \vdash \mathbf{1}}\ (!\mathrm{weak}) \qquad \dfrac{\dfrac{!A \vdash !A \quad !A \vdash !A}{!A, !A \vdash !A \otimes !A}\ (\otimes_r)}{!A \vdash !A \otimes !A}\ (!\mathrm{cont}) \quad \leftrightarrow \quad \begin{matrix} w^!_A : [\![!A]\!] \to [\![\mathbf{1}]\!] \\ c^!_A : [\![!A]\!] \to [\![!A]\!] \otimes [\![!A]\!] \end{matrix}$$

$*$-Autonomous Categories: Adding the Exponentials

- Seely noted the categorical content of the exponential in 1989 [29]:
  - $\to$! is given by a *comonad* on a $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$

$$! : \mathbb{X} \to \mathbb{X} \qquad p^!_A : !A \to !!A \quad (!\text{-}\mathbf{digging}) \qquad d^!_A : !A \to A \quad (!\text{-}\mathbf{dereliction})$$

  - $\to$ $!A$ has a comonoid structure

$$c^!_A : !A \to !A \otimes !A \quad (!\text{-}\mathbf{contraction}) \qquad w^!_A : !A \to \mathbf{1} \quad (!\text{-}\mathbf{weakening})$$

  - $\to$ natural isomorphisms, known as the **Seely isomorphisms**

$$!A \otimes !B \cong !(A \,\&\, B) \qquad !\top \cong \mathbf{1}$$

- Altogether, this implies that *coKleisli* category $\mathbb{X}_!$ is *cartesian closed*: categorical model for intuitionistic logic.
  - $\to$ Key aspect of the exponential ! is the *intuitionistic implication decomposition* from LL:

$$A \Rightarrow B = !A \multimap B$$

- Bierman demonstrated that Seely's model was not quite right in 1995 [7].
  - $\to$ Interpretation of proofs in Seely's models is not necessarily invariant under cut-elimination.

- Alternative models were introduced, without assuming additives, generally for intuitionistic LL.
- First categorical models for intuitionistic MELL were given by Lafont in 1988.
- Given a symmetric monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$, we denote the category of cocomm comonoids and comonoid morphisms in $\mathbb{X}$ by $\mathrm{C}[\mathbb{X}]$.

Definition (Lafont [24])

A **Lafont category** is a symmetric monoidal (closed) category $(\mathbb{X}, \otimes, \mathbf{1})$, wherein the forgetful functor $\mathrm{U} : \mathrm{C}[\mathbb{X}] \to \mathbb{X}$ has a right adjoint

$$\mathrm{U} \dashv \,! : \mathrm{C}[\mathbb{X}] \to \mathbb{X}$$

Alternatively, for each $A \in \mathbb{X}$, there is a cocomm $\otimes$-comonoid $!A$ and a map $d^!_A$

$$c^!_A : !A \to !A \otimes !A \qquad w^!_A : !A \to \mathbf{1} \qquad d^!_A : !A \to A$$

satisfying the following universal property:

for every cocomm. $\otimes$-comonoid $\langle X, \Delta_X, e_X \rangle$ and map $f : X \to A$, there is a unique comonoid map $f^\flat : \langle X, \Delta_X, e_X \rangle \to \langle !A, c^!_A, w^!_A \rangle$ such that

$$X \xrightarrow{f^\flat} !A \xrightarrow{d^!_A} A = X \xrightarrow{f} A$$

- Adapted to the classical linear logic context:

  a $*$-*autonomous* Lafont category $(\mathbb{X}, \otimes, \mathbf{1}, *, !)$ is a model for MELL
- MELL: de Morgan dualities relate ! and ?

Proposition (Folklore)

Given a $*$-*autonomous Lafont category* $(\mathbb{X}, \otimes, \mathbf{1}, *, !)$, *define the why not modality structure by the following:*

$$?A = (!(A^*))^* \qquad c^?_A : ?A \parr ?A \cong (!(A^*) \otimes !(A^*))^* \xrightarrow{(c^!_{A^*})^*} (!(A^*))^* = ?A$$

$$w^?_A : \perp = \mathbf{1}^* \xrightarrow{(w^!_{A^*})^*} (!(A^*))^* = ?A \qquad d^?_A : A \cong A^{**} \xrightarrow{(d^!_{A^*})^*} (!A^*)^* = ?A$$

Then, the forgetful functor $\mathrm{U} : \mathrm{M}[\mathbb{X}] \to \mathbb{X}$ *has a left adjoint*

$$? \dashv \mathrm{U} : \mathbb{X} \to \mathrm{M}[\mathbb{X}]$$

In other words, ? *is the cofree comm.* $\parr$*-monoid construction.*

- This definition requires ! is the **free** cocomm $\otimes$-comonoid construction, which *leaves* out examples of models for MELL.

- The now *accepted* definition for a *general* model of intuitionistic MELL was first proposed by Benton, Bierman, de Paiva and Hyland in 1992 [5].

Definition (Bierman [7])

A **linear category** is a symmetric monoidal (closed) category $(\mathbb{X}, \otimes, \mathbf{1})$, equipped with

- a symmetric monoidal comonad $(!, p^!, d^!, \mu^!, \mu_\mathbf{1})$, and

$$! : \mathbb{X} \to \mathbb{X} \qquad p^!_A : !A \to !!A \quad (!\text{-}\textbf{digging}) \qquad d^!_A : !A \to A \quad (!\text{-}\textbf{dereliction})$$

$$\mu^!_{A,B} : !A \otimes !B \to !(A \otimes B) \qquad \mu_\mathbf{1} : \mathbf{1} \to !\mathbf{1}$$

- monoidal natural transformations

$$c^!_A : !A \to !A \otimes !A \quad (!\text{-}\textbf{contraction}) \qquad w^!_A : !A \to \mathbf{1} \quad (!\text{-}\textbf{weakening})$$

such that

1. $(!A, c^!_A, w^!_A)$ is a cocommutative comonoid,
2. $c^!_A$ and $w^!_A$ are !-coalgebra morphisms, and
3. $p^!_A$ is a comonoid morphism.

- The comonad structure $(!, p^!, d^!, \mu^!, \mu_1, c^!, w^!)$ has been given other names:
  - *linear exponential comonad* by Hyland and Schalk [23]
  - *storage modality* by Blute, Cockett and Seely [11]
  - *monoidal coalgebra modality* by Blute, Cockett, Lemay and Seely [9]

$\Rightarrow$ Categorical models for MELL (as remarked on by Hyland and Schalk [23]):

$*$**-autonomous categories** equipped with **monoidal coalgebra modalities,** otherwise known as $*$**-autonomous linear categories**

- Key property of these models is the induced **linear-non-linear adjunction**:

Proposition (Benton [6])

Given a model for MELL *as above, then consider the coE-M category of coalgebras* $\mathbb{X}^!$*:*

$$\textit{objects } (A, \nu_A : A \to !A) \qquad \textit{maps } f : A \to B \textit{ such that } f; \nu_B = \nu_A; f$$

$\mathbb{X}^!$ *is cartesian closed and there is a symmetric monoidal adjunction* $\mathrm{U} \dashv ! : \mathbb{X}^! \to \mathbb{X}$.

$\rightarrow$ coKleisli category $\mathbb{X}_!$ is equivalent to a full subcategory of coE-M category $\mathbb{X}^!$ consisting of **free** co-algebras

$$(!A, p^!_A : !A \to !!A)$$

- MELL: de Morgan dualities relate ! and ?

Proposition (Folklore)

Given a $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ equipped with monoidal coalgebra modality $(!, p^!, d^!, \mu^!, \mu_{\mathbf{1}}, c^!, w^!)$, define the why not modality structure by the following:

$$? : \mathbb{X} \to \mathbb{X} \qquad A \mapsto (!(A^*))^*$$

$$p^?_A : ??A \cong (!!(A^*))^* \xrightarrow{(p^!_{A^*})^*} (!(A^*))^* = ?A \quad (?\text{-}\textbf{digging})$$

$$d^?_A : A \cong A^{**} \xrightarrow{(d^!_{A^*})^*} (!A^*)^* = ?A \quad (?\text{-}\textbf{dereliction})$$

$$\mu^?_{A,B} : ?(A \parr B) \cong (!(A^* \otimes B^*))^* \xrightarrow{(\mu^!_{A^*,B^*})^*} (!(A^*) \otimes !(B^*))^* \cong ?A \parr ?B$$

$$\mu_\perp : ?\perp \cong (!\mathbf{1})^* \xrightarrow{\mu^*_{\mathbf{1}}} \mathbf{1}^* = \perp$$

$$c^?_A : ?A \parr ?A \cong (!(A^*) \otimes !(A^*))^* \xrightarrow{(c^!_{A^*})^*} (!(A^*))^* = ?A \quad (?\text{-}\textbf{contraction})$$

$$w^?_A : \perp = \mathbf{1}^* \xrightarrow{(w^!_{A^*})^*} (!(A^*))^* = ?A \quad (?\text{-}\textbf{weakening})$$

Then, $(?, p^?, d^?, \mu^?, \mu_\perp, c^?, w^?)$ is a monoidal coalgebra modality on the symmetric monoidal category $(\mathbb{X}^{op}, \parr, \perp)$.

Example

1. Every $*$-autonomous Lafont category $(\mathbb{X}, \otimes, \mathbf{1}, *, !)$ is a $*$-autonomous linear category [27].
2. **Rel**: the category sets and relations $\rightarrow$ $*$-autonomous Lafont category

$$!X = \mathrm{M}_{fin}(X) = \{[\![x_1, \ldots x_n]\!]\} \quad \text{(set of all finite multisets of elements in } X\text{)}$$

$$!R = \{([\![x_1, \ldots x_n]\!], [\![y_1, \ldots y_n]\!]) \mid (x_i, y_i) \in R, \forall i\}$$

3. **Chu construction** (Hyland and Schalk [23])

   Consider a symmetric closed monoidal category $(\mathbb{X}, \otimes, \mathbf{1})$ with pullbacks, and with **well-adapted monoids** and **a suitable strength**, and an object $\perp \in \mathbb{X}$, then $\mathrm{Chu}(\mathbb{X}, \perp)$ is a $*$-autonomous linear category.

Remark

The details of the definition of well-adapted monoids and a suitable strength are said to be in a companion paper, but I was unable to find it.

- From the perspective of MELL^+ and models based on SLDCs:
  $\rightarrow$! and ? must both be taken as primitive.
- This was developed by Blute, Cockett and Seely in 1996 [10].
- The definition was made more succinct by Cockett and Seely in 1999:

Definition (Cockett, Seely [15])

A SLDC $(\mathbb{X}, \otimes, \mathbf{1}, \parr, \perp)$ **has storage** if it is equipped with

- a monoidal coalgebra modality $(!, p^!, d^!, \mu^!, \mu_{\mathbf{1}}, c^!, w^!)$ on $(\mathbb{X}, \otimes, \mathbf{1})$, and
- a monoidal coalgebra modality $(?, p^?, d^?, \mu^?, \mu_\perp, c^?, w^?)$ on $(\mathbb{X}^{op}, \parr, \perp)$,

such that $(!, ?)$ is a symmetric linear functor, and $(p^!, p^?)$, $(d^!, d^?)$, $(c^!, c^?)$ and $(w^!, w^?)$ are linear transformations.

The key to the above definition are the *linear strengths*:

$$v^R_{\otimes A,B} : !(A \parr B) \to ?A \parr !B \qquad v^R_{\parr A,B} : !A \otimes ?B \to ?(A \otimes B)$$

$$v^L_{\otimes A,B} : !(A \parr B) \to !A \parr ?B \qquad v^L_{\parr A,B} : ?A \otimes !B \to ?(A \otimes B)$$

⇒ Categorical models for MELL^+ (as shown by Blute, Cockett and Seely [10]):

SLDCs equipped with a **LD-coalgebra modalities**,
otherwise known as **SLDCs with storage**

Example

1. Every $*$-autonomous category $(\mathbb{X}, \otimes, \mathbf{1}, *)$ equipped with a monoidal coalgebra modality $(!, p^!, d^!, \mu^!, \mu_\mathbf{1}, c^!, w^!)$ is a storage SLDC, with why not modality structure given by de Morgan duality [15]:

$$?A = (!(A^*))^*$$

2. **More examples?**

⇒ Categorical models for MELL (as shown by Cockett and Seely [15]):

$*$**-autonomous linear categories**

$\Leftrightarrow$

storage SLDCs with **negation**

Categorical Models for LL and LL^{+}

- In 1995, Bierman introduced a modification to Seely's formulation, solving the issue: **new Seely categories** [7].
- New Seely categories subsequently received equivalent reformulations:
  - *Seely categories* by Melliès [27]
  - *tensor storage categories* by Blute, Cockett and Seely [8].

Definition (Melliès [27])

A **Seely category** is a symmetric monoidal (closed) category $(\mathbb{X}, \otimes, \mathbf{1})$ with finite products $(\&, \top)$, equipped with

- a comonad $(!, p^!, d^!)$, and
- natural Seely isomorphisms

$$n^2_{A,B} : !A \otimes !B \cong !(A \& B) \qquad n_{\mathbf{1}} : \mathbf{1} \cong !\top$$

such that $(!, n^2, n_{\mathbf{1}}) : (\mathbb{X}, \&, \top) \to (\mathbb{X}, \otimes, \mathbf{1})$ is a symmetric lax monoidal functor and

$$\begin{array}{ccccc}
!A \otimes !B & & \xrightarrow{\quad p^!_A \otimes p^!_B \quad} & & !!A \otimes !!B \\
\downarrow {\scriptstyle n^2_{A,B}} & & & & \downarrow {\scriptstyle n^2_{!A,!B}} \\
!(A \& B) & \xrightarrow[p^!_{A\&B}]{} & !!(A \& B) & \xrightarrow[!\langle !\pi^0_{A,B}, !\pi^1_{A,B} \rangle]{} & !(!A \& !B)
\end{array}$$

Models for $\mathrm{LL}$and LL^+: Altogether Now

- Bierman showed Seely categories are models for LL.
- Considering our models for MELL and MALL, we expect a model for LL should be a $*$-autonomous linear category with finite products.
- Bierman showed these are equivalent:

Proposition (Bierman [7])

Every Seely category is a linear category and every linear category with finite products is a Seely category.

$\Rightarrow$ Categorical models for LL (as shown by Bierman [7], Cockett and Seely [15]):

$*$**-autonomous linear categories** with **finite products**

$\Leftrightarrow$

$*$**-autonomous Seely categories**

$\Leftrightarrow$

storage SLDCs with **negation** and **finite linear products**

$\Rightarrow$ Categorical models for LL^+ (as shown by Cockett and Seely [15]):

storage SLDCs with **finite linear products**

Thank you for your attention!

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