Linear Turing Categories: Their coKleisli and coEilenberg-Moore Categories, the Differential Case, and Examples

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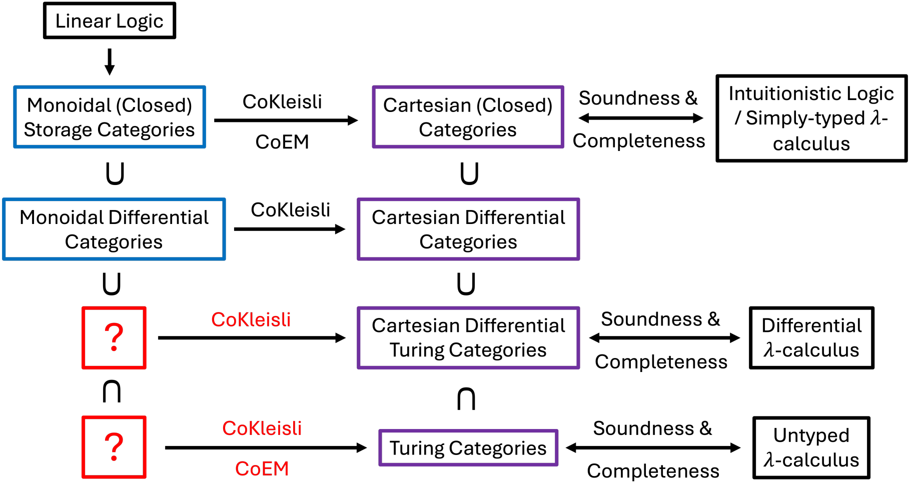
A Bird’s Eye View



Untyped λ-Calculus

The **untyped λ-calculus** is like the simply typed λ-calculus, where the term formation rules application and abstraction are replaced by

$$\frac{\Gamma \vdash m \colon U \qquad \Gamma \vdash n \colon U}{\Gamma \vdash mn \colon U}\,(\text{App.}) \qquad\qquad \frac{\Gamma, x \colon U, \Gamma' \vdash m \colon U}{\Gamma, \Gamma' \vdash \lambda x.m \colon U}\,(\text{Abs.})$$

Turing Categories (TCs)

Definition 1

A **Turing category**[a] $\mathbb{X}$ is a Cartesian category with

1. a distinguished object T, called a **Turing object**, and
2. for each B, C, a morphism $T \times B \xrightarrow{\bullet_{B,C}} C$, called **application**,

such that for each $A \times B \xrightarrow{f} C$, $\exists$ a **code** $A \xrightarrow{c_f} T$ of f such that

$$\begin{tikzcd}
T \times B \arrow[r, "\bullet"] & C \\
A \times B \arrow[u, "c_f \times \mathrm{id}_B"] \arrow[ur, "f"'] &
\end{tikzcd}$$

[a] Note that there is a more general notion of a Turing category defined on a **restriction category**. Here, we are focusing only on the so-called **total** case.

Canonical Codes for TCs

Definition 2

A Turing category $\mathbb{X}$ has **canonical codes** when there exists a function $\lambda_{A,B,C} \colon \mathbb{X}(A \times B, C) \to \mathbb{X}(A, T)$ for each triple of objects A, B, C such that the diagrams

$$\begin{tikzcd} T \times B \arrow[r, "\bullet_{B,C}"] & C \\ A \times B \arrow[u, "\lambda_{A,B,C}[f] \times \mathrm{id}_B"] \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} & T \\ D \arrow[ur, "\lambda_{D,B,C}[(g \times \mathrm{id}_B);f]"] \arrow[r, "g"'] & A \arrow[u, "\lambda(f)"'] \end{tikzcd}$$

commute for all $A \times B \xrightarrow{f} C$ and $D \xrightarrow{g} A$.

What is Linear Logic?

Chemistry tells us that: $H + H + O \to H_2O$

If Chem Eqs. had contraction: $H + O \to H_2O$

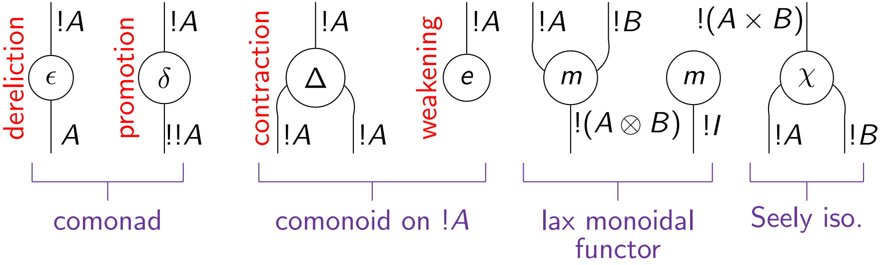
If Chem Eqs. had weakening: $H + H + O + C \to H_2O$

Linear logic is a roughly logic where normal types, like A, DON'T have contraction & weakening, but types of the form $!A$ do!

Monoidal Storage Categories (MSCs)

A **monoidal storage category** is a symmetric monoidal category with

1. biproducts,
2. an endofunctor $!$, and
3. the following natural transformations:



such that

1. Δ_A and e_A are $!$-coalgebra morphisms,[1]
2. δ_A is a comonoid morphism.

[1]Note that $(!A, \delta_A)$ is a $!$-coalgebra, as $(!, \delta, \epsilon)$ is a comonad.

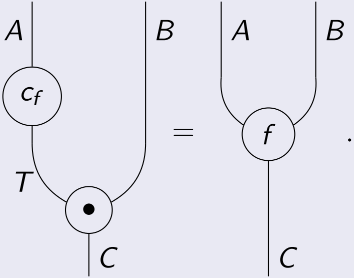
Linear Turing Categories (LTCs)

Definition 3

A **linear Turing category** $\mathbb{X}$ is a monoidal storage category with

1. a distinguished object T, called a **Turing object**, and
2. for each B, C, a morphism $T \otimes B \xrightarrow{\bullet_{B,C}} C$, called **application**,

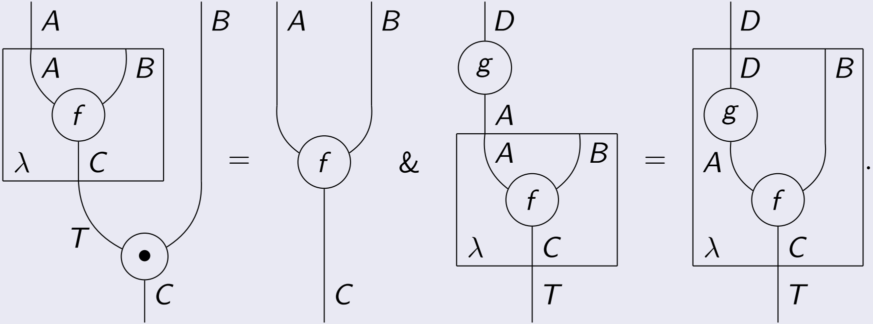
s.t. for each $A \otimes B \xrightarrow{f} C$, $\exists$ a **code** of $A \xrightarrow{c_f} T$ of f s.t.



Canonical Codes for LTCs

Definition 4

A linear Turing category $\mathbb{X}$ has **canonical codes** when $\exists$ a function $\lambda_{A,B,C} \colon \mathbb{X}(A \otimes B, C) \to \mathbb{X}(A, T)$ for each triple of objects A, B, C s.t.



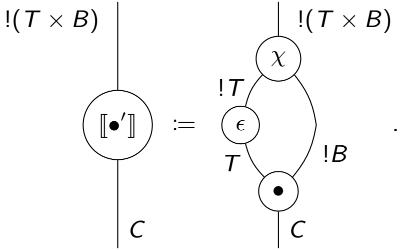
for all $A \otimes B \xrightarrow{f} C$ and $D \xrightarrow{g} A$.

The Co-Kleisli Category of an LTC (Part 1/3)

Theorem 5

Let $\mathbb{X}$ be a LTC (with canonical codes). Then, the co-Kleisli category $\mathbb{X}_!$ is a Turing category with Turing object T (and canonical codes).

Proof: For objects B, C, define the application $\bullet'_{B,C} \colon T \times B \to C$ by



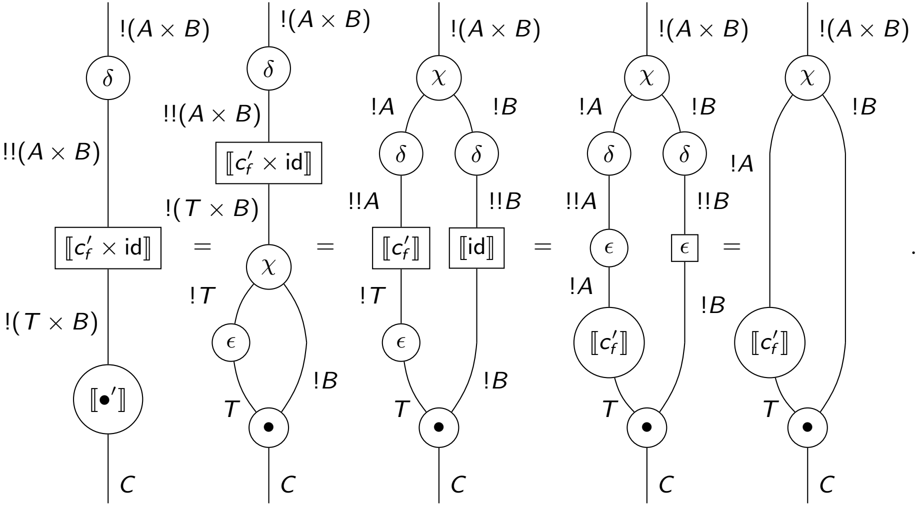
Take any $f \colon A \times B \to C$ in $\mathbb{X}_!$. Define the (canonical) $\mathbb{X}_!$-code c'_f of f by

$$[\![c'_f]\!] := \,!A \xrightarrow{c_{\chi^{-1}_{A,B};[\![f]\!]}} T,$$

where $c_{\chi^{-1}_{A,B};[\![f]\!]}$ is a (canonical) $\mathbb{X}$-code of $!A \otimes !B \xrightarrow{\chi^{-1}_{A,B}} !(A \times B) \xrightarrow{[\![f]\!]} C$.

The Co-Kleisli Category of an LTC (Part 2/3)

Proof (Cont.): Then, we have the usual currying relation.



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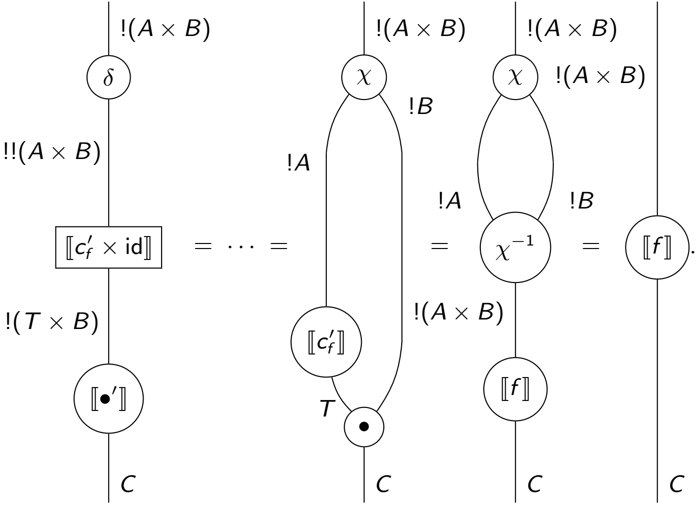
The Co-Kleisli Category of an LTC (Part 3/3)

Proof (Cont.): By the definition of c_f,

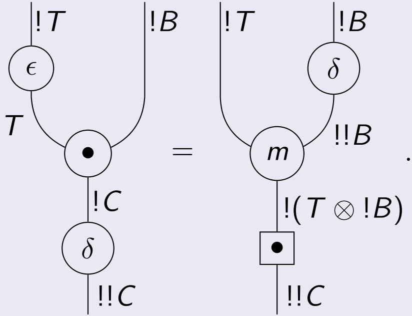


Note that the LHS is $[\![c'_f \times \mathrm{id}_B ; \bullet'_{B,C}]\!]$, whence $c'_f \times \mathrm{id}_B ; \bullet'_{B,C} = f$ in $\mathbb{X}_!$.

Neat Linear Turing Categories (NLTCs)

Definition 6

A linear Turing category $\mathbb{X}$ is called **neat** if for each pair of objects B, C,



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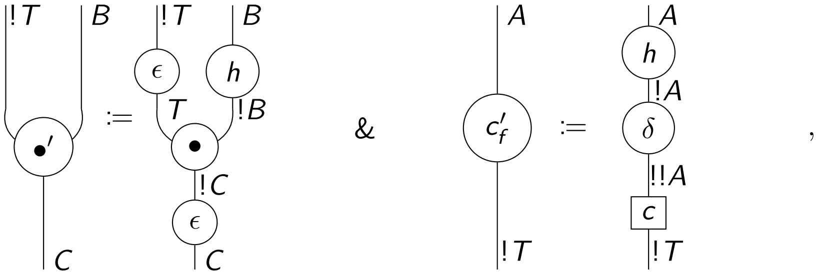
That is, $!T \otimes !B \xrightarrow{\epsilon_T \otimes \mathrm{id}_{!B}} T \otimes !B \xrightarrow{\bullet_{!B,!C}} !C$ is a coalgebra morphism.

Theorem 7

Let $\mathbb{X}$ be a NLTC (with canonical codes). Then, the coEM category $\mathbb{X}^!$ is a Turing category with Turing object $!T$ (and canonical codes).

Proof: Define the application and code of $\mathbb{X}^!$ as follows:

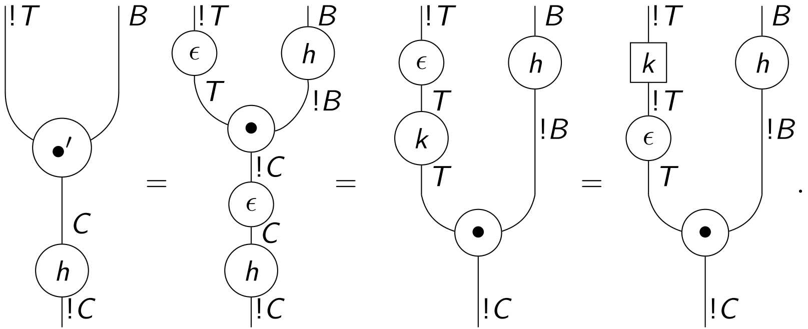
$$\forall f : (A, h_A) \otimes (B, h_B) \to (C, h_C),$$



where c is a (canonical) $\mathbb{X}$-code of $!A \otimes !B \xrightarrow{m_{A,B}} !(A \otimes B) \xrightarrow{!f} !C$.

The Co-Eilenberg-Moore Category of a NLTC (Part 2/3)

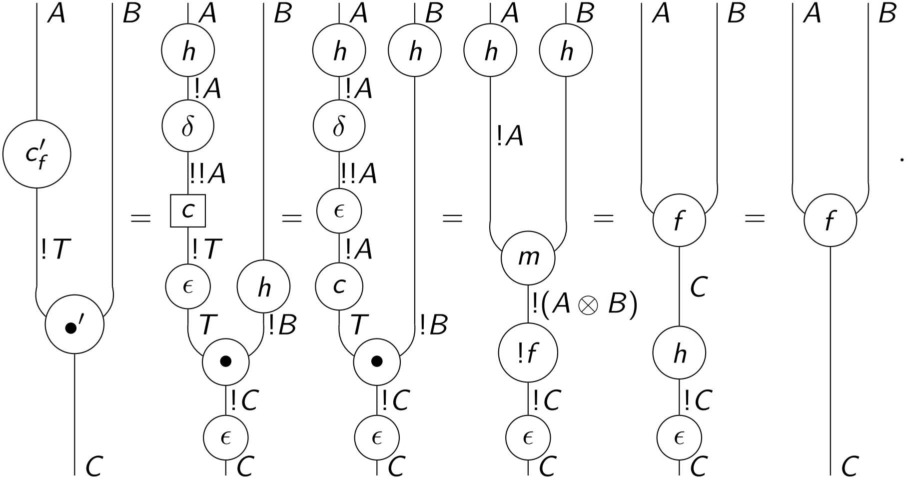
Proof [$\bullet'$ is a coalg. morph. in $\mathbb{X}$]: Take any obj. $(B, h_B), (C, h_C)$ in $\mathbb{X}^!$. Since $\mathbb{X}$ is an LTC, $\exists$ a code k of $T \otimes !B \xrightarrow{\bullet_{B,!C}} !C \xrightarrow{\epsilon_C} C \xrightarrow{h_C} !C$. Then,



- Since the RHS is a coalg. morph., the LHS, $\bullet'_{B,C}; h_C$, is also.
- Since h_C is a coalg. morph. & a section, $\bullet'_{B,C}$ is a coalg. morph.

The Co-Eilenberg-Moore Category of a NLTC (Part 3/3)

Proof [$\bullet'$ is an application map in $\mathbb{X}^!$]: Take any coalgebra morphism $(A, h_A) \otimes (B, h_B) \xrightarrow{f} (C, h_C)$ in $\mathbb{X}$. Then,



Cartesian Differential Categories (CDCs)

A **Cartesian differential category** $\mathbb{X}$ is a left additive Cartesian category equipped with a **differential combinator** D, which is a fam. of operators:

$$D\colon \mathbb{X}(A,B) \to \mathbb{X}(A \times A, B) \qquad (f\colon A \to B) \mapsto (D[f]\colon A \times A \to B),$$

where $D[f]$ is called the **derivative of** f, such that a num. of axioms holds.

Intuition: If we think of a morphism in $\mathbb{X}$ as a function $f : \mathbb{R}^m \to \mathbb{R}^n$, then

$$D[f](\mathbf{x}, \mathbf{v}) \text{ behaves like } J_f(\mathbf{x}) \cdot \mathbf{v}.$$

Cartesian Differential Categories (CDTCs)

Definition 8

A Turing category $\mathbb{X}$ is a **Cartesian differential Turing category** when

1. it is also a Cartesian differential category and
2. its application map $\bullet_{B,C} \colon T \times B \to C$ is linear in its 1st arg. $\forall B, C$.

Definition 9

A CDTC $\mathbb{X}$ has **additive canonical codes** when $\mathbb{X}$ has canon. codes s.t.

1. $\forall f, g \colon A \times B \to C$ in $\mathbb{X}$, $\lambda_{A,B,C}[f + g] = \lambda_{A,B,C}[f] + \lambda_{A,B,C}[g]$;
2. $\forall A, B, C$ in $\mathbb{X}$, $\lambda_{A,B,C}[0_{A \times B,C}] = 0_{A,T}$.

Definition 10

A CDTC $\mathbb{X}$ has **differential canonical codes** when $\mathbb{X}$ has add. codes s.t.

- $\forall f \colon A \times B \to C$, $A \times A \xrightarrow{D\left[A \xrightarrow{\lambda[f]} T\right]} T := \lambda\left[(A \times A) \times B \xrightarrow{D_1[f]} C\right]$.

Monoidal Differential Categories (MDCs)

A **monoidal differential category** $\mathbb{X}$ is a monoidal storage category equipped with a natural transformation

$$\eta : A \to !A,$$

called **codereliction**, satisfying a num. of axioms corresponding to diff.

Importantly, the following diagram commutes for all objects A:

$$\begin{tikzcd} !A \arrow[r, "\epsilon_A"] & A \\ A \arrow[u, "\eta_A"] \arrow[ur, equal] & \end{tikzcd} .$$

While it doesn't immediately look like it, this actually corresponds to the fact that the derivative of a linear function is a constant!

Monoidal Differential Turing Categories (MDTCs)

Definition 11

A LTC $\mathbb{X}$ is called a **monoidal differential Turing category** when it is also a monoidal differential category.

Definition 12

A MDTC $\mathbb{X}$ has **additive canonical codes** when $\mathbb{X}$ has canon. codes s.t.

1. $\forall f, g \colon A \otimes B \to C$ in $\mathbb{X}$, $\lambda_{A,B,C}[f + g] = \lambda_{A,B,C}[f] + \lambda_{A,B,C}[g]$;
2. $\forall A, B, C$ in $\mathbb{X}$, $\lambda_{A,B,C}[0_{A \otimes B,C}] = 0_{A,T}$.

Theorem 13

Let $\mathbb{X}$ be an MDTC (with *additive* canonical codes). Then, $\mathbb{X}_!$ is a CDTC (with *differential* canonical codes).

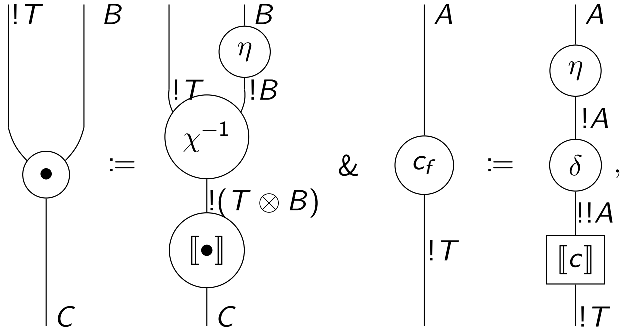
Proof: A lot of diagrams :(

This is the ONLY Answer (Part 1/2)

Theorem 14

Let $\mathbb{X}$ be a MDC. Then, $\mathbb{X}_!$ is a CDTC iff $\mathbb{X}$ is a LTC.

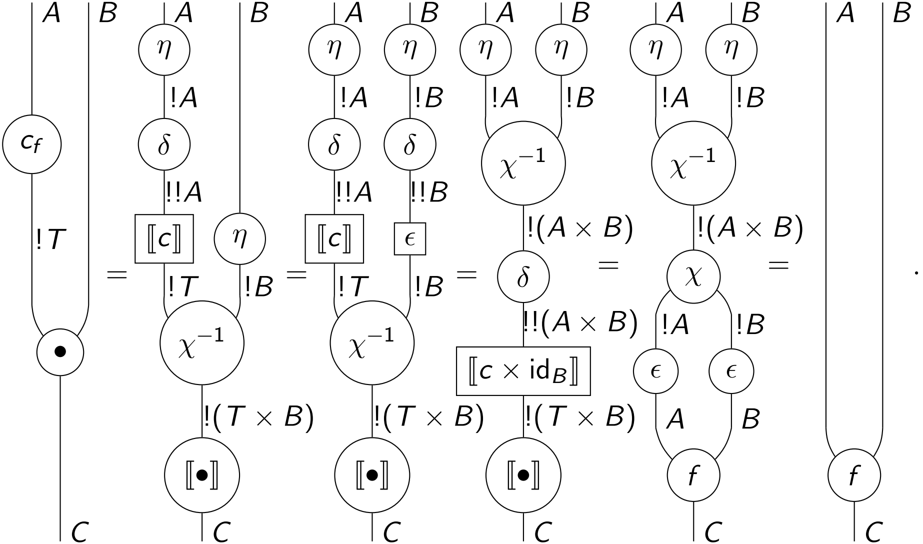
Proof ($\Leftarrow$): If $\mathbb{X}$ is an LTC, it's an MDTC, so by **T.13**, $\mathbb{X}_!$ is a CDTC.
Proof ($\Rightarrow$): Take any $f\colon A \otimes B \to C$. Def. the $\mathbb{X}$-app. & $\mathbb{X}$-code of f by



where c is the $\mathbb{X}_!$-code of $!(A \times B) \xrightarrow{\chi_{A,B}} !A \otimes !B \xrightarrow{\epsilon_A \otimes \epsilon_B} A \otimes B \xrightarrow{f} C$.

This is the ONLY Answer (Part 2/2)

Proof (⇒) (Continued): Then, the usual currying relation holds.



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The 1st Kleene Algebra induces a MDTC

Example 15

Let $\mathbb{K} = (\mathbb{N}, \cdot)$ be the 1st Kleene Algebra, where $m \cdot n = \phi_m(n)$, and let $\mathbf{Rel}_{\mathbb{K}}$ be the subcategory of $\mathbf{Rel}$ such that the objects are $\mathbb{N}^n$ for some $n \in \mathbb{N}$ and $R \colon \mathbb{N}^m \to \mathbb{N}^n$ is a morphism when $\exists r \in \mathbb{N}$ such that

$$R = \{(a_1, \ldots, a_{m+n}) \in \mathbb{N}^{m+n} : r \cdot a_1 \cdots a_{m+n} \downarrow\}.$$

Then, $\mathbf{Rel}_{\mathbb{K}}$ is a MDTC.

Proof of currying equation: Define the application on $\mathbb{N}$ by

$$\bullet \coloneqq \{(a, b, c) \in \mathbb{N}^3 : a \cdot b \cdot c \downarrow\}.$$

Take any $R \colon \mathbb{N} \otimes \mathbb{N} \to \mathbb{N}$ in $\mathbf{Rel}_{\mathbb{K}}$. Then, $\exists r \in \mathbb{N}$ such that

$$R = \{(a, b, c) \in \mathbb{N}^3 : r \cdot a \cdot b \cdot c \downarrow\}.$$

Define the code of R by

$$c_R \coloneqq \{(a, r \cdot a) \in \mathbb{N}^2\}.$$

Then, $c_R \otimes \mathrm{id}_{\mathbb{N}}; \bullet = R$.