Mini-Course: Category Theory in Topological Data Analysis

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Categories

- A *category* $\mathbf{C}$ is a collection of objects, $\mathbf{C}_0$, along with *morphisms* between those objects.
- The collection of morphisms from x to y in $\mathbf{C}_0$ we will denote by $\mathbf{C}(x, y)$.
- Morphisms are composable whenever it makes sense. This composition is associative, and each object has an *identity morphism* that is neutral with respect to composition.

The Standard Examples

- **Set**: sets and mappings
- $\mathbf{Vec}_{\mathbb{k}}$: vector spaces (over a given field $\mathbb{k}$) and linear transformations
- $\mathbf{vec}_{\mathbb{k}}$: finite-dimensional vector spaces and linear transformations
- **Top**: topological spaces and continuous maps

Important for TDA: Preordered sets

- A *proset* is a set P along with a relation $\leq$ that is
  - reflexive: $x \leq x$ for all $x \in P$
  - transitive: if $x \leq y$ and $y \leq z$ then $x \leq z$.
- We often identify the proset $(P, \leq)$ with the category with objects P, and precisely one morphism from x to y whenever $x \leq y$ (otherwise none).

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- We often identify the proset $(P, \leq)$ with the category with objects P, and precisely one morphism from x to y whenever $x \leq y$ (otherwise none).
- (Posets are evil.)

Another important one: Relations

- The category **Rel** has, as objects, all sets.
- If A and B are sets, then $\mathbf{Rel}(A, B)$ consists of all relations from A to B, that is, all subsets $S \subseteq A \times B$.
- Composition: if $S \in \mathbf{Rel}(A, B)$ and $T \in \mathbf{Rel}(B, C)$, then

$$T \circ S = \{(a, c) \in A \times C : \exists b \in B, (a, b) \in S, (b, c) \in T\}.$$

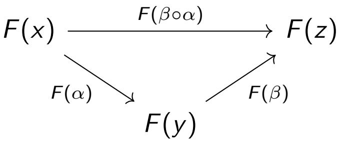
- The identity relation on A is the diagonal of $A \times A$, i.e., equality.
- **Set** is a subcategory of **Rel**.

Comparing Categories: Functors

Let **A** and **C** be categories.

- A *functor* $F : \mathbf{A} \to \mathbf{C}$ consists of
  - a map $F_0 : \mathbf{A}_0 \to \mathbf{C}_0$, and
  - for each $x, y \in \mathbf{A}_0$, a mapping $F : \mathbf{A}(x,y) \to \mathbf{C}(F(x), F(y))$; the image of $\alpha : x \to y$ is denoted $F(\alpha)$,

  such that
  - F preserves identities: $F(1_x) = 1_{F(x)}$;
  - F preserves composition: the diagram



    commutes.

Persistence modules

Let **D** be any category. A functor $F : (\mathbb{R}, \leq) \to \mathbf{D}$ is called a *persistence module*. It consists of:

- for each $a \in \mathbb{R}$, an object $F(a)$;
- whenever $a \leq b$, a morphism $F_{a \leq b} : F(a) \to F(b)$; these morphisms satisfy the composition rule

  $$F_{a \leq c} = F_{b \leq c} \circ F_{a \leq b}$$

  whenever $a \leq b \leq c$.

Persistence modules and sub-level sets

Let us specialize to $\mathbf{D} = \mathbf{Top}$. (Can specialize further to topological spaces and inclusions.) Let $f : X \to \mathbb{R}$ be a function on the topological space X.

- For $a \in \mathbb{R}$, set $F(a) = f^{-1}((-\infty, a])$.
- If $a \leq b$ then $(-\infty, a] \subseteq (-\infty, b]$, so $F(a) \hookrightarrow F(b)$; easy to see functorial.
- Apply $H_k(-; \mathbb{k})$ to get

  $$H \circ F : (\mathbb{R}, \leq) \to \mathbf{vec}_{\mathbb{k}}$$

  (if X finite type).

Comparing Functors: Natural Transformations

Let $F, G : \mathbf{A} \to \mathbf{C}$ be functors. A *natural transformation* $\alpha : F \Rightarrow G$ consists of, for each $a \in \mathbf{A}$, a morphism in $\mathbf{C}$, $\alpha_a : F(a) \to G(a)$, such that for every morphism $\varphi : a \to a'$ in $\mathbf{A}$, the diagram

$$\begin{array}{ccc} F(a) & \xrightarrow{\alpha_a} & G(a) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(a') & \xrightarrow[\alpha_{a'}]{} & G(a') \end{array}$$

commutes.

Diagram Categories

Let $\mathbf{A}$ and $\mathbf{C}$ be categories, where the objects of $\mathbf{A}$ form a set. The collection of all functors $F : \mathbf{A} \to \mathbf{C}$ comprise the objects of a category, denoted by $\mathbf{C}^{\mathbf{A}}$, with natural transformations as morphisms. If $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$, then their (horizontal) composition is defined componentwise by $(\beta \circ \alpha)_a = \beta_a \circ \alpha_a$ for all $a \in \mathbf{A}$.

Example: Translations

We consider the poset $(\mathbb{R}, \leq)$.

- Let $\varepsilon \geq 0$. *Translation by* ε is the function defined by $T_\varepsilon(x) = x + \varepsilon$.
- Since $T_\varepsilon(x) \leq T_\varepsilon(y)$ whenever $x \leq y$, translation is in fact an endofunctor on $(\mathbb{R}, \leq)$.
- Since, for all $x \in \mathbb{R}$, $x \leq T_\varepsilon(x)$, we get a natural transformation $\eta : I \Rightarrow T_\varepsilon$, where I is the identity functor on $\mathbb{R}$.

Interleavings

(Chazal, Cohen-Steiner, Glisse, Guibas, Oudot 2009)

Let $\varepsilon \geq 0$.

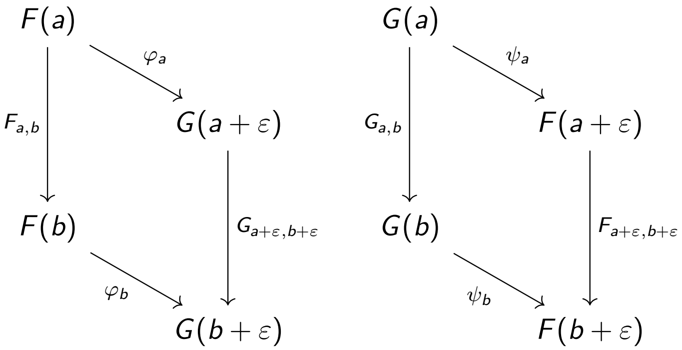
- For any persistence module $F : (\mathbb{R}, \leq) \to \mathbf{C}$, the composite $F \circ T_\varepsilon$ is a “shifted” version of F.
- We would like to compare two modules, $F, G : (\mathbb{R}, \leq) \to \mathbf{C}$. The idea we use is that of *interleaving*.
- Interleaving is a generalization of isomorphism (not quite an equivalence relation, though).
- Will define original interleavings, then generalize.

“Classic” interleavings

- $F, G : (\mathbb{R}, \leq) \to \mathbf{C}$ are ε-*interleaved* if there exist natural transformations $\varphi : F \to G \circ T_\varepsilon$ and $\psi : G \to F \circ T_\varepsilon$, such that
- $\psi \circ \varphi = F \circ \eta_{2\varepsilon}$ and $\varphi \circ \psi = G \circ \eta_{2\varepsilon}$.
- We should unpack this definition (to get the original).

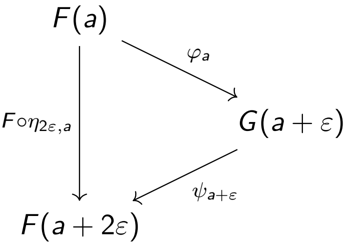
Interleavings continued

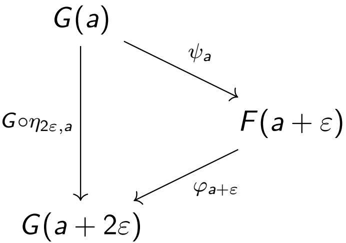
The following diagrams commute for all $a \leq b$:



Interleavings continued

The following diagrams commute for all $a \in \mathbb{R}$:





Example

Let I be any interval in $\mathbb{R}$. Let $\mathbb{k}_I : (\mathbb{R}, \leq) \to \textbf{vec}$ be the "characteristic" persistence module for I:

- $\mathbb{k}_I(a) = \mathbb{k}_I$ if $a \in I$, otherwise $\mathbb{k}_I(a) = 0$.
- If $a \leq b$, and $a, b \in I$, then $(\mathbb{k}_I)_{a,b} = 1_{\mathbb{k}}$.

If I has length $< 2\varepsilon$, then $\mathbb{k}_I$ is ε-interleaved with the zero module.

Generalizing interleavings and Future Equivalences

Let $\mathbf{P}$ and $\mathbf{Q}$ be small categories. Consider functors $F : \mathbf{P} \to \mathbf{C}$ and $G : \mathbf{Q} \to \mathbf{C}$. The key to determining the proximity of F and G is a notion from directed homotopy theory, namely, *future equivalence*.

Future Equivalences

(Grandis 2005)
A future equivalence from **P** to **Q** consists of a quadruple, $(\Gamma, \mathrm{K}, \eta, \nu)$, where

- $\Gamma : P \to Q$ and $\mathrm{K} : Q \to P$ are functors,
- $\eta : \mathrm{I}_P \Rightarrow \mathrm{K}\Gamma$ and $\nu : \mathrm{I}_Q \Rightarrow \Gamma\mathrm{K}$ are natural transformations, and
- we have the coherence conditions,

$$\Gamma\eta = \nu\Gamma : \Gamma \Rightarrow \Gamma\mathrm{K}\Gamma \quad \text{and} \quad \mathrm{K}\nu = \eta\mathrm{K} : \mathrm{K} \Rightarrow \mathrm{K}\Gamma\mathrm{K}.$$

Interleavings of Functors

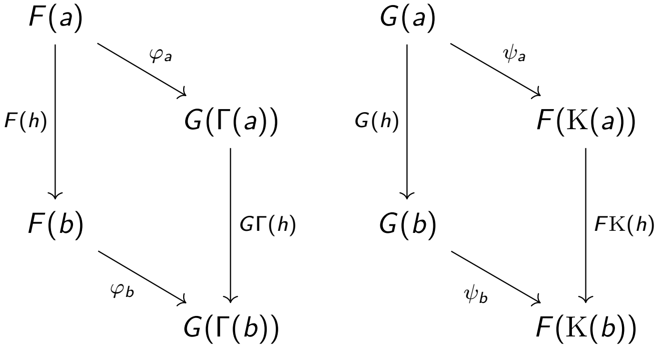
Let $(\Gamma, \mathrm{K}, \eta, \nu)$ be a future equivalence from **P** to **Q**. We say that functors $F : \mathbf{P} \to \mathbf{C}$ and $G : \mathbf{Q} \to \mathbf{C}$ are $(\Gamma, \mathrm{K}, \eta, \nu)$-*interleaved* if there exist natural transformations

$$\varphi : F \Rightarrow G\Gamma \quad \text{and} \quad \psi : G \Rightarrow F\mathrm{K}$$

such that $\psi_\Gamma \varphi = F\eta$ and $\varphi_\mathrm{K} \psi = G\nu$.

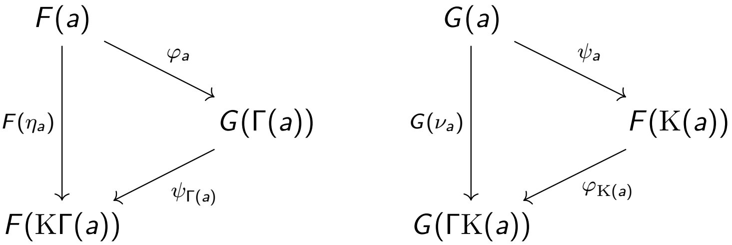
Unpacking the Definitions

We get a similar bunch of diagrams that need to commute. Whenever there is a morphism $h : a \to b$:



Still Unpacking

For all $a \in \mathbf{P}$:



Dynamical Systems

- A *discrete dynamical system* is a topological space X along with a continuous self-map $f : X \to X$.

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- A *discrete dynamical system* is a topological space X along with a continuous self-map $f : X \to X$.
- From our categorical point of view, we consider a dynamical system to be a functor $F : N \to \textbf{Top}$, where N is the category with one object x and morphisms φ^k for $k \geq 0$, $F(x) = X$ and $F(\varphi) = f$.

Shift Equivalences

Dynamical systems $f : X \to X$ and $g : Y \to Y$ are said to be *shift equivalent with lag* ℓ if there exist continuous maps $\alpha : X \to Y$ and $\beta : Y \to X$ such that $\alpha f = g\alpha$, $\beta g = f\beta$, $\beta\alpha = f^\ell$, and $\alpha\beta = g^\ell$.

Exercises

1. What are the possible functors, $\Gamma : N \to N$?
2. If $\Gamma, \mathrm{K} : N \to N$ and $\alpha : \Gamma \Rightarrow \mathrm{K}$, what are the possibilities for the component α_x, and what does the existence of α say about Γ and K?
3. Show that if there exists $\eta : \mathrm{I} \Rightarrow \Gamma\mathrm{K}$, then $\Gamma = \mathrm{K} = \mathrm{I}$.

The future equivalences of the "dynamical system category" are all in the natural transformations, not the translations!

Solutions

1. $\Gamma(x) = x$, $\Gamma(\varphi) = \varphi^k$ for some $k \geq 0$.
2. We must have $\alpha_x = \varphi^m$ for some $m \geq 0$. If $\Gamma(\varphi) = \varphi^k$ and $\mathrm{K}(\varphi) = \varphi^\ell$, then the diagram

   $$\begin{array}{ccc} x & \xrightarrow{\alpha_x} & x \\ {\scriptstyle \Gamma(\varphi)}\downarrow & & \downarrow{\scriptstyle \mathrm{K}(\varphi)} \\ x & \xrightarrow{\alpha_x} & x \end{array}$$

   implies that $k + m = \ell + m$, so $k = \ell$, so $\Gamma = \mathrm{K}$.
3. From the previous exercise, $\Gamma\mathrm{K} = \mathrm{I}$, from which it follows that $\Gamma = \mathrm{K} = \mathrm{I}$.

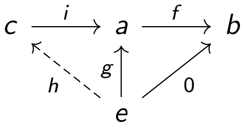
Abelian Categories

A category **A** is *abelian* if:

- hom (morphism) sets are abelian groups, and composition is biadditive;
- finite direct sums and direct products exist and the natural morphism
$$a \oplus b \to a \times b$$
is an isomorphism;
- every morphism has a kernel and a cokernel;
- every monomorphism is the kernel of some morphism; every epimorphism is the cokernel of some morphism.

Kernels (and cokernels)

Let $\mathbf{A}$ be an abelian category. For any $a, b \in \mathbf{A}$, we have a *zero morphism* $0 : a \to b$. Let $f : a \to b$ be any morphism. We say that $i : c \to a$ is the *kernel* of f if whenever the right triangle commutes, there is a unique $h : e \to c$ making the left triangle commute.



We usually abuse notation and write $c = \ker f$.
To get the definition of cokernels, we reverse arrows.

Exercise: Kernels in **Rel**

The category **Rel** turns out to be important (Edelsbrunner *et al* 2015, Bauer-Lesnick 2019) in studying the partial matchings of persistence *diagrams* required for calculating the bottleneck distance.

Rel is not abelian, but it does have zero morphisms and kernels.

1. What is $0 \in \mathbf{Rel}(X, Y)$?
2. Let $R \subseteq \mathbf{Rel}(X, Y)$. Find the kernel of R.

Solutions

1. $0 = \varnothing \subseteq X \times Y$.
2. The kernel of R is the subset K of unmatched elements of X; the "inclusion" is the "full" relation $K \times X$.

The Category of Interleavings

There is a category, $\mathbf{Int}_\varepsilon$, in which the objects are ε-interleaved pairs of persistence modules $F, G : (\mathbb{R}, \leq) \to \mathbf{C}$ and morphisms are pairs of natural transformations that make the appropriate diagrams commute.

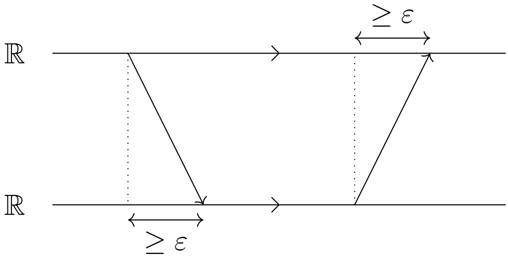
- If $\mathbf{C}$ is abelian, is $\mathbf{Int}_\varepsilon$ abelian? Yes!
- Vin de Silva saw our tedious direct proof in Bubenik-S. (2014) and was mortified.

Interleavings Form a Diagram Category

- Vin observed that $\mathbf{Int}_\varepsilon$ is itself a diagram category, and it is a standard exercise to show that if $\mathbf{A}$ is abelian and $\mathbf{D}$ is small, then $\mathbf{A}^{\mathbf{D}}$ is abelian. (Everything is computed pointwise.)

Interleavings Form a Diagram Category

- Vin observed that $\mathbf{Int}_\varepsilon$ is itself a diagram category, and it is a standard exercise to show that if $\mathbf{A}$ is abelian and $\mathbf{D}$ is small, then $\mathbf{A}^{\mathbf{D}}$ is abelian. (Everything is computed pointwise.)
- The indexing category, I_ε, looks like this:



Remarks on Interleavor Categories

- The category I_ε turns out to be a Grothendieck construction, that is, a certain pullback of categories.
- The construction works even for future equivalences of pairs of small categories.
- Eventually leads to a Gromov-Hausdorff metric on the category of (weighted) small categories (Bubenik, de Silva, S., 2016)

Metrics

- The whole point of interleavings is to give a metric on persistence modules: we say that $d_I(F, G) \leq \varepsilon$ if there exists an ε-interleaving between F and G.
- More generally, if $F : \mathbf{P} \to \mathbf{C}$ and $G : \mathbf{Q} \to \mathbf{C}$ are (Γ, K)-interleaved, we need to have some sort of measure of the translations Γ and K (or the pair).
- Will restrict our attention to the case where $\mathbf{P}$ is a proset, $\mathbf{Q} = \mathbf{P}$.

Sublinear Projections

Let **P** be a proset. A *sublinear projection* is a function, $\omega : \mathbf{Trans_P} \to [0, \infty]$ such that

- $\omega_{\mathrm{I}} = 0$, where I is the identity translation.
- $\omega_{\Gamma \mathrm{K}} \le \omega_{\Gamma} + \omega_{\mathrm{K}}$.

Example on $(\mathbb{R}, \le)$:

$$\omega_{\Gamma} = \sup\{\Gamma(x) - x : x \in \mathbb{R}\}$$

Distance Associated to a Sublinear Projection

Let ω be a sublinear projection on the preordered set $\mathbf{P}$.

1. Γ is an ε-*translation* if $\omega_\Gamma \leq \varepsilon$.
2. $F, G : \mathbf{P} \to \mathbf{C}$ are ε-*interleaved* if F and G are (Γ, K)-interleaved for some pair of ε-translations, Γ and K.
3. interleaving distance:

$$d^\omega(F, G) = \inf\{\varepsilon \geq 0 : F, G \ \varepsilon\text{-interleaved w.r.t. } \omega\}$$

Superlinear Families

Let $\mathbf{P}$ be a proset.

- A *superlinear family* on $\mathbf{P}$ is a function, $\Omega : [0, \infty) \to \mathbf{Trans_P}$ such that $\Omega_{\varepsilon_1}\Omega_{\varepsilon_2} \geq \Omega_{\varepsilon_1+\varepsilon_2}$.
- example on $(\mathbb{R}, \leq)$: $\Omega_\varepsilon : t \mapsto t + \varepsilon$ (called this T_ε earlier).
- example on poset of subsets of a metric space X: the ε-offset of a subset,

$$A^\varepsilon = \{x \in X : d(x, A) \leq \varepsilon\}$$

- $d^\Omega(F, G) = \inf\{\varepsilon : F, G \text{ are } \Omega_\varepsilon\text{-interleaved}\}$.

A Theorem

(Bubenik, de Silva, S., 2014) Let ω be a sublinear projection on a preordered set $\mathbf{P}$. Suppose for every $\varepsilon \geq 0$ there exists a translation Ω_ε with $\omega_{\Omega_\varepsilon} \leq \varepsilon$, which is 'largest' in the sense that $\omega_\Gamma \leq \varepsilon$ implies $\Gamma \leq \Omega_\varepsilon$. Then $\varepsilon \mapsto \Omega_\varepsilon$ is a superlinear family, and the two interleaving distances are the same.

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More succinctly: ω can be regarded as a functor. If ω has a *right adjoint* Ω, then Ω is a superlinear family that yields the same distance function.

Exercises

1. Verify that
$$\omega_\Gamma = \sup\{\Gamma(x) - x : x \in \mathbb{R}\}$$
defines a sublinear projection on $(\mathbb{R}, \leq)$.
2. Verify that
$$A \mapsto A^\varepsilon$$
defines a superlinear family on $\mathbf{P}_X$, the poset of subsets of the metric space X.

Some Further Directions

- categories with flow (de Silva, Munch, Stefanou 2018)
- Kan extensions: used to extend maps from subspaces of metric spaces (Bubenik,de Silva, Nanda 2017)
- generalized persistence diagrams: Patel 2016
- erosion distance (an interleaving-type metric for persistence diagrams): Puuska 2017