The Poincaré Lemma for Codifferential Categories with Antiderivatives

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CMS Summer 2019

Thanks to CMS for student funding support.

Poincaré Lemma

For an open subset $U \subseteq \mathbb{R}^n$, let $\Omega^*(U)$ be the **de Rham complex** of U.

- $\Omega^n(U) := \mathcal{C}^\infty(U) \otimes \bigwedge^k \mathbb{R}^n$ is the set of n forms
- δ is the exterior derivative with $\delta \circ \delta = 0$
- **Closed**: $\delta(\omega) = 0$, that is, $\omega \in \ker(\delta)$
- **Exact**: $\omega = \delta(\nu)$, that is, $\omega \in \mathrm{im}(\delta)$
- $\mathrm{im}(\delta) \subset \ker(\delta)$ and so exact $\Rightarrow$ closed

Theorem

For a contractible open subset $U \subseteq \mathbb{R}^n$, $\Omega^*(U)$ is **contractible**, *that is, homotopy equivalent to the zero complex or equivalently $id_{\Omega(U)}$ is homotopic to* 0.

$$s : \Omega^{k+1}(U) \to \Omega^k(U) \qquad \delta\left(s(\omega)\right) + s\left(\delta(\omega)\right) = \omega$$

Therefore every closed form is exact, that is, $\mathrm{im}(\delta) = \ker(\delta)$. *In particular, $\Omega^*(\mathbb{R}^n)$ is contractible.*

TODAY'S STORY: Generalize the Poincaré Lemma for codifferential categories.

Codifferential Categories - Blute, Cockett, Seely (2006)

A **codifferential category** consists of:

- A (strict) symmetric monoidal category $(\mathbb{X}, \otimes, I, \sigma)$, which is enriched over commutative monoids: so each hom-set is a commutative monoid with an addition operation $+$ and a zero 0, such that the additive structure is preserves by composition and $\otimes$.

- An **algebra modality**, which is a monad (T, μ, η)

  $$\mu : \mathsf{TT}(A) \to \mathsf{T}(A) \qquad \eta : A \to \mathsf{T}(A)$$

  equipped with two natural transformations:

  $$\mathsf{m} : \mathsf{T}(A) \otimes \mathsf{T}(A) \to \mathsf{T}(A) \qquad \mathsf{u} : I \to \mathsf{T}(A)$$

  such that $\mathsf{T}(A)$ is a commutative monoid and μ is a monoid morphism.

- And equipped with a **deriving transformation**, which is a natural transformation:

  $$\mathsf{d} : \mathsf{T}(A) \to \mathsf{T}(A) \otimes A$$

  which satisfies certain equalities which encode the basic properties of differentiation.

R. Blute, R. Cockett, R.A.G. Seely, **Differential Categories**, Mathematical Structures in Computer Science Vol. 1616, pp 1049-1083, 2006.

Example: Smooth Functions

Example

A $\mathcal{C}^\infty$-**ring** is commutative $\mathbb{R}$-algebra A such that for each for smooth map $f : \mathbb{R}^n \to \mathbb{R}$ there is a function $\Phi_f : A^n \to A$ and such that the Φ_f satisfy certain coherences between them.

Ex. For a smooth manifold M, $\mathcal{C}^\infty(M) = \{f : M \to \mathbb{R} | \ f \text{ smooth}\}$ is a $\mathcal{C}^\infty$-ring.

There is an adjunction:

$$\mathsf{VEC}_{\mathbb{R}} \underset{U}{\overset{\mathsf{T}^\infty}{\rightleftarrows}} \mathcal{C}^\infty\mathsf{Ring} \quad (\mathsf{T}^\infty \dashv U)$$

The induced monad is an algebra modality and has a deriving transformation.

In particular, $\mathsf{T}^\infty(\mathbb{R}^n) = \mathcal{C}^\infty(\mathbb{R}^n)$, and so μ and η correspond to composition of smooth functions, while m and u correspond to multiplication of smooth functions. And the deriving transformation is:

$$\mathsf{d} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n$$
$$f \longmapsto \sum_i \frac{\partial f}{\partial x_i} \otimes x_i$$

So $\mathsf{VEC}_{\mathbb{R}}$ is a codifferential category, that is, $\mathsf{VEC}_{\mathbb{R}}^{op}$ is a differential category.

Cruttwell, G.S.H., Lemay, J.S. and Lucyshyn-Wright, R.B.B., 2019. **Integral and differential structure on the free C^∞-ring modality**. arXiv preprint arXiv:1902.04555.

de Rham complex in codifferential categories

Our next step is to build the de Rham complex for $\mathsf{T}(A)$ is suitable codifferential categories.

O'Neill, K., 2017. Smoothness in codifferential categories (PhD Thesis).

Assume that we are working in a codifferential category which is enriched over $\mathbb{Q}$-modules (negatives and rationals!) and has split idempotents: so that we can build exterior powers!

de Rham complex in codifferential categories

Let Σ_n be the set of n permutations. Then for each object A we obtain an idempotent p_n:

$$A \otimes \ldots \otimes A \xrightarrow{\mathsf{p}_n := \frac{1}{n!} \cdot \sum\limits_{\tau \in \Sigma_n} \mathsf{sgn}(\tau) \cdot \tau} A \otimes \ldots \otimes A$$

Then for an object A, define its nth exterior power $\bigwedge\limits^{n} A$ as the following idempotent splitting:

$$\begin{array}{ccc} \bigotimes\limits^{n} A & \xrightarrow{\mathsf{p}_n} & \bigotimes\limits^{n} A \\ & {\scriptstyle \mathsf{m}_n} \searrow \quad \nearrow {\scriptstyle \mathsf{r}_n} & \\ & \bigwedge\limits^{n} A & \end{array} \qquad \begin{array}{ccc} \bigwedge\limits^{n} A & = & \bigwedge\limits^{n} A \\ & {\scriptstyle \mathsf{r}_n} \searrow \quad \nearrow {\scriptstyle \mathsf{m}_n} & \\ & \bigotimes\limits^{n} A & \end{array}$$

By convention, $\bigwedge\limits^{0} A := I$ and $\bigwedge\limits^{1} A := A$.

Example

In $\mathsf{VEC}_{\mathbb{R}}$,

$$\mathsf{m}_2(v \otimes w) = v \wedge w \qquad \mathsf{r}_2(v \wedge w) = \frac{1}{2} \cdot v \otimes w - \frac{1}{2} \cdot w \otimes v$$

de Rham complex in codifferential categories

For each object A, the **de Rham complex** of $\mathsf{T}(A)$ is defined as follows:

$$K \xrightarrow{\mathsf{u}} \mathsf{T}(A) \xrightarrow{\mathsf{d}} \mathsf{T}(A) \otimes A \xrightarrow{\delta} \mathsf{T}(A) \otimes \bigwedge^{2} A \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathsf{T}(A) \otimes \bigwedge^{n} A \xrightarrow{\delta} \ldots$$

where $\delta : \mathsf{T}(A) \otimes \bigwedge^{n} A \to \mathsf{T}(A) \otimes \bigwedge^{n+1} A$ is the exterior derivative and is defined as:

$$\delta := \mathsf{T}(A) \otimes \bigwedge^{n} A \xrightarrow{\mathsf{d} \otimes \mathsf{r}_n} \mathsf{T}(A) \otimes A \otimes \underbrace{A \otimes \ldots \otimes A}_{n-\text{times}} \xrightarrow{1 \otimes \mathsf{m}_{n+1}} \mathsf{T}(A) \otimes \bigwedge^{n+1} A$$

And we have that $\delta\delta = 0$

GOAL: To show that the de Rham complex is contractible:

$$\mathsf{T}(A) \otimes \bigwedge^{n+1} A \xrightarrow{\zeta} \mathsf{T}(A) \otimes \bigwedge^{n} A \qquad \zeta\delta + \delta\zeta = 1$$

For this we need **antiderivatives**

Antiderivatives

Cockett, J.R.B. and Lemay, J.S., 2019. **Integral categories and calculus categories**. Mathematical Structures in Computer Science, 29(2), pp.243-308.

In a codifferential category, define the natural transformation $\mathsf{L} : \mathsf{T}(A) \to \mathsf{T}(A)$ as follows:

$$\mathsf{L} := \ \mathsf{T}(A) \xrightarrow{\quad \mathsf{d} \quad} \mathsf{T}(A) \otimes A \xrightarrow{\quad 1 \otimes \eta \quad} \mathsf{T}(A) \otimes \mathsf{T}(A) \xrightarrow{\quad \mathsf{m} \quad} \mathsf{T}(A)$$

A codifferential category has **antiderivatives** if the natural transformation $\mathsf{K} : \mathsf{T}(A) \to \mathsf{T}(A)$

$$\mathsf{K} := \mathsf{L} + \mathsf{T}(0)$$

is a natural isomorphism.

Define the **integral transformation** $\mathsf{s} : \mathsf{T}(A) \otimes A \to \mathsf{T}(A)$ as follows:

$$\mathsf{s} := \ \mathsf{T}(A) \otimes A \xrightarrow{\quad 1 \otimes \eta \quad} \mathsf{T}(A) \otimes \mathsf{T}(A) \xrightarrow{\quad \mathsf{m} \quad} \mathsf{T}(A) \xrightarrow{\quad \mathsf{K}^{-1} \quad} \mathsf{T}(A)$$

In particular, the deriving transformation and integral transformation are compatible via the fundamental theorems of calculus – more on this soon!

Antiderivatives - Examples

Example

$\mathsf{VEC}_{\mathbb{R}}$ is a codifferential category with antiderivatives.

For a smooth map $f : \mathbb{R}^n \to \mathbb{R}$:

$$\mathsf{K} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad \mathsf{K}[f](\vec{v}) = \nabla(f)(\vec{v}) \cdot \vec{v} + f(\vec{0})$$

$$\mathsf{K}^{-1} : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad \mathsf{K}^{-1}[f](\vec{v}) = \int_0^1 \int_0^1 \nabla(f)(st\vec{v}) \cdot \vec{v} \; \mathsf{d}s \; \mathsf{d}t + f(\vec{0})$$

$$\mathsf{s} : \mathcal{C}^\infty(\mathbb{R}^n) \otimes \mathbb{R}^n \to \mathcal{C}^\infty(\mathbb{R}^n) \qquad \mathsf{s}(f \otimes e_i)(\vec{v}) = \int_0^1 f(t\vec{v}) v_i \; \mathsf{d}t$$

Antiderivatives

A codifferential category has **antiderivatives** if K is a natural isomorphism.

Define the **integral transformation** $\mathsf{s} : \mathsf{T}(A) \otimes A \to \mathsf{T}(A)$ as follows:

$$\mathsf{s} := \mathsf{T}(A) \otimes A \xrightarrow{1\otimes\eta} \mathsf{T}(A) \otimes \mathsf{T}(A) \xrightarrow{\mathsf{m}} \mathsf{T}(A) \xrightarrow{\mathsf{K}^{-1}} \mathsf{T}(A)$$

The deriving transformation and integral transformation are compatible via the fundamental theorems of calculus.

- **Second Fundamental Theorem of Calculus**:

$$\mathsf{ds} + \mathsf{T}(0) = 1 \qquad \int_0^x \frac{\partial f(u)}{\partial u}(t)\ \mathrm{d}t + f(0) = f(x)$$

- **Poincaré Condition**: If $f : B \to \mathsf{T}(A) \otimes A$ is such that

$$f(\mathsf{d} \otimes 1)(1 \otimes \sigma) = f(\mathsf{d} \otimes 1)$$

  then f satisfies **the First Fundamental Theorem**:

$$f\mathsf{sd} = f$$

  This says that closed 1-forms are exact: without negatives!

Contractible de Rham from Antiderivatives

Let's build our contraction with antiderivatives:

$$K \underset{\zeta}{\overset{\mathsf{u}}{\rightleftarrows}} \mathsf{T}(A) \underset{\zeta}{\overset{\mathsf{d}}{\rightleftarrows}} \mathsf{T}(A) \otimes A \underset{\zeta}{\overset{\delta}{\rightleftarrows}} \mathsf{T}(A) \otimes \bigwedge^{2} A \underset{\zeta}{\overset{\delta}{\rightleftarrows}} \dots \underset{\zeta}{\overset{\delta}{\rightleftarrows}} \mathsf{T}(A) \otimes \bigwedge^{n} A \underset{\zeta}{\overset{\delta}{\rightleftarrows}} \dots$$

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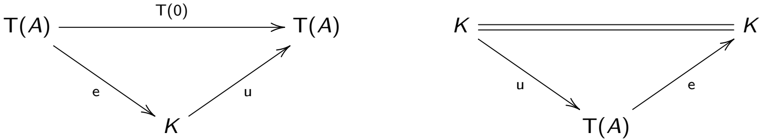
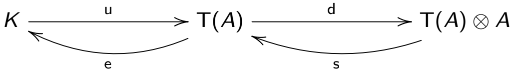
Splitting T(0)

Notice that $\mathsf{T}(0) : \mathsf{T}(A) \to \mathsf{T}(A)$ is an idempotent. We require this splits via K, that is, there is a natural transformation $\mathsf{e} : \mathsf{T}(A) \to K$ which makes $\mathsf{T}(A)$ into an augmented monoid:



Then by the Second Fundamental Theorem of Calculus, we have that:



$$\mathsf{ds} + \mathsf{eu} = \mathsf{ds} + \mathsf{T}(0) = 1 \qquad \mathsf{ue} + 0 = 1$$

Contractible de Rham from Antiderivatives

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$$K \xrightarrow{\mathsf{u}} \mathsf{T}(A) \xrightarrow{\mathsf{d}} \mathsf{T}(A) \otimes A \xrightarrow{\delta} \mathsf{T}(A) \otimes \bigwedge^{2} A \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathsf{T}(A) \otimes \bigwedge^{n} A \xrightarrow{\delta} \dots$$

$$K \xleftarrow{\mathsf{e}} \mathsf{T}(A) \xleftarrow{\mathsf{s}} \mathsf{T}(A) \otimes A \xleftarrow{\zeta} \mathsf{T}(A) \otimes \bigwedge^{2} A \xleftarrow{\zeta} \dots \xleftarrow{\zeta} \mathsf{T}(A) \otimes \bigwedge^{n} A \xleftarrow{\zeta} \dots$$

So what we want is:

$$\mathsf{T}(A) \otimes \bigwedge^{n+1} A \xrightarrow{\zeta} \mathsf{T}(A) \otimes \bigwedge^{n} A \qquad \zeta\delta + \delta\zeta = 1$$

Contractible de Rham from Antiderivatives

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First attempt:

$$\mathsf{T}(A) \otimes \bigwedge^{n+1} A \xrightarrow{1 \otimes \mathsf{r}_{n+1}} \mathsf{T}(A) \otimes A \otimes \underbrace{A \otimes \dots \otimes A}_{n} \xrightarrow{\mathsf{s} \otimes \mathsf{m}_n} \mathsf{T}(A) \otimes \bigwedge^n A$$

Contractible de Rham from Antiderivatives

Let's build our contraction with antiderivatives:

$$K \underset{e}{\overset{u}{\rightleftarrows}} T(A) \underset{s}{\overset{d}{\rightleftarrows}} T(A)\otimes A \underset{\zeta}{\overset{\delta}{\rightleftarrows}} T(A)\otimes \bigwedge^2 A \underset{\zeta}{\overset{\delta}{\rightleftarrows}} \cdots \underset{\zeta}{\overset{\delta}{\rightleftarrows}} T(A)\otimes \bigwedge^n A \underset{\zeta}{\overset{\delta}{\rightleftarrows}} \cdots$$

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$$T(A)\otimes \bigwedge^{n+1} A \xrightarrow{\zeta} T(A)\otimes \bigwedge^{n} A \qquad \zeta\delta + \delta\zeta = 1$$

First attempt:

$$T(A)\otimes \bigwedge^{n+1} A \xrightarrow{1\otimes r_{n+1}} T(A)\otimes A \otimes \underbrace{A\otimes \ldots \otimes A}_{n} \xrightarrow{s\otimes m_n} T(A)\otimes \bigwedge^{n} A$$

THIS DOES NOT WORK!

Example

$$\delta\left(\zeta\left(xy\otimes(x\wedge y)\right)\right) + \delta\left(\zeta\left(xy\otimes(x\wedge y)\right)\right) = \frac{2}{3}\cdot xy\otimes(x\wedge y)$$

The lettre J is here to save the day!

Define the following family of natural transformations $\mathsf{J}_n : \mathsf{T}(A) \to \mathsf{T}(A)$:

$$\mathsf{J}_0 := \mathsf{L} \qquad \mathsf{J}_{n+1} := \mathsf{J}_n + 1$$

Theorem (Cockett and Lemay)

In a codifferential category with antiderivatives, for every $n \in \mathbb{N}$, J_{n+1} is a natural isomorphism.

Proof: By induction. For $n = 0$, J_1^{-1} is defined as follows:

$$\mathsf{T}(A) \xrightarrow{1 \otimes \mathsf{u}} \mathsf{T}(A) \otimes K \xrightarrow{\mathsf{T}(\eta) \otimes \eta} \mathsf{TT}(A) \otimes \mathsf{TT}(A) \xrightarrow{\mathsf{m}} \mathsf{TT}(A) \xrightarrow{\mathsf{K}^{-1}} \mathsf{TT}(A) \xrightarrow{\mu} \mathsf{T}(A)$$

Assuming J_{n+1} is an isomorphism, J_{n+2}^{-1} is defined as follows:

$$\mathsf{T}(A) \xrightarrow{1 \otimes \mathsf{u}} \mathsf{T}(A) \otimes K \xrightarrow{\mathsf{T}(\eta) \otimes \eta} \mathsf{TT}(A) \otimes \mathsf{TT}(A) \xrightarrow{\mathsf{m}} \mathsf{TT}(A) \xrightarrow{\mathsf{J}_{n+1}^{-1}} \mathsf{TT}(A) \xrightarrow{\mu} \mathsf{T}(A)$$

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First define $\mathsf{s}_{n+1} : \mathsf{T}(A)\otimes A \to \mathsf{T}(A)$ as follows:

$$\mathsf{s}_{n+1} := \ \mathsf{T}(A)\otimes A \xrightarrow{1\otimes\eta} \mathsf{T}(A)\otimes\mathsf{T}(A) \xrightarrow{\mathsf{m}} \mathsf{T}(A) \xrightarrow{\mathsf{J}_{n+1}^{-1}} \mathsf{T}(A)$$

by convention $\mathsf{s}_0 := \mathsf{s}$.

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by convention $\mathsf{s}_0 := \mathsf{s}$.

Define the contraction $\zeta : \mathsf{T}(A) \otimes \bigwedge^{n+1} A \to \mathsf{T}(A) \otimes \bigwedge^{n} A$ as follows:

$$\mathsf{T}(A) \otimes \bigwedge^{n+1} A \xrightarrow{1 \otimes \mathsf{r}_{n+1}} \mathsf{T}(A) \otimes A \otimes \underbrace{A \otimes \ldots \otimes A}_{n} \xrightarrow{\mathsf{s}_n \otimes \mathsf{m}_n} \mathsf{T}(A) \otimes \bigwedge^{n} A$$

And this works!

Theorem

In a codifferential category with antiderivatives, enriched over $\mathbb{Q}$-modules, and the necessary idempotent splitting, the de Rham complex of $\mathsf{T}A$ is contractible with contraction ζ.

$$\delta\zeta + \zeta\delta = 1 \qquad \zeta\zeta = 0$$

Last Few Words

- This results is also true for infinite dimensional vector spaces!
- One can take other examples of codifferential categories. For example, taking $\mathsf{T} = \mathsf{Sym}$, this gives the algebraic version of the Poincaré lemma, i.e, that the de Rham complex of Kahler differentials for polynomial rings (over arbitrary sets) is contractible.

  Hartshorne, R., 1975. **On the De Rham cohomology of algebraic varieties**. Publications Mathematiques de l'IHES, 45, pp.5-99.

- What does the de Rham complex mean for differential categories/differential linear logic?
- In a codifferential category: It is possible to build the de Rham complex for any T-algebra. So what can we say about T-algebras whose de Rham complex is contractible?

  (For example the T^{∞}-algebra $\mathbb{C}^{\infty}(M)$, for some contractible smooth manifold M.)
- This is an example of a graded **Rota-Baxter** algebra: the integral counterpart to graded differential algebras. (which I don't think these have been studied...)

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Thanks for listening! Merci!

Go Raptors!