The equivalence of ordered groupoids and left cancellative categories using double categories[1]

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[1] Joint work with Dorette Pronk

Ordered groupoids

Definition

An *ordered groupoid* is a category $\mathcal{G}$ such that

1. All arrows are invertible;
2. There is a partial order relation on the arrows which extends to the objects via the identity arrows;
3. The order is preserved by taking inverses and composition.
4. When $f : A \to B$ and $A' \leq A$ there is a unique restriction $[f|_* A'] : A' \to B'$ such that $[f|_* A'] \leq f$,

$$\begin{array}{ccc} A' & \xrightarrow{[f|_* A']} & B' \\ \leq & \leq & \leq \\ A & \xrightarrow[f]{} & B \end{array}$$

If $B' \leq B$, we similarly have unique corestrictions written as $[B'_*|f]$.

Ordered groupoids as internal categories

We can also view this as an internal groupoid

$$\mathcal{G}_1 \times_{\mathcal{G}_0} \mathcal{G}_1 \xrightarrow{m} \mathcal{G}_1 \xrightarrow{i} \mathcal{G}_1 \underset{s}{\overset{t}{\underset{\xleftarrow{u}}{\rightrightarrows}}} \mathcal{G}_0$$

in the category of partially ordered sets with an additional property corresponding to the last requirement given above: the domain arrow $\mathcal{G}_1 \xrightarrow{s} \mathcal{G}_0$ is a fibration as a functor between posetal categories.

Ordered groupoids as double categories

Another way to view this last diagram is as a double category $\mathcal{G}$ where the vertical arrows give the poset structure and the horizontal arrows give the groupoid structure. Double cells have the following form

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ \leq \big\downarrow & \leq & \big\downarrow \leq \\ X' & \xrightarrow[g']{} & Y' \end{array} \tag{1}$$

Here, $X \leq X'$, $Y \leq Y'$ and $g \leq g'$.Note that in this notation, the fact that $s\colon \mathcal{G}_1 \to \mathcal{G}_0$ is a fibration corresponds to the statement that for each diagram

$$\begin{array}{ccc} X & & \\ \leq \big\downarrow & & \\ X' & \xrightarrow[g']{} & Y' \end{array}$$

there is a unique diagram (1).

Ordered groupoids and left cancellative categories

Lawson introduced a constructions moving between between ordered groupoids and left cancellative categories.

lcCat is the category of left cancellative categories with functors as morphisms.

oGpd is the category of ordered groupoids as above with double functors as morphisms.

The functor $\mathbf{L}$

The functor $\mathbf{L} \colon \mathbf{oGpd} \to \mathbf{lcCat}$ is defined as follows.

For an ordered groupoid $\mathcal{G}$, the left cancellative category $\mathbf{L}(\mathcal{G})$ has as objects those of $\mathcal{G}$.

An arrow $A \to B$ in $\mathbf{L}(\mathcal{G})$ is a formal composite of a horizontal arrow in $\mathcal{G}$ with a vertical arrow in $\mathcal{G}$: they are of the form

$$A \xrightarrow{h} B' \overset{\leq}{\longrightarrow} B$$

where h is a horizontal arrow in $\mathcal{G}$ and $B' \overset{\leq}{\longrightarrow} B$ is a vertical arrow in $\mathcal{G}$.

The functor **L**

Composition uses the restriction operation in $\mathcal{G}$:

$$\begin{array}{ccccc}
A & \xrightarrow{\;h\;} & B' & \xrightarrow{\;[k|_* B']\;} & C'' \\
 & & {\scriptstyle\le}\,\downarrow\bullet & \le & \bullet\downarrow\,{\scriptstyle\le} \\
 & & B & \xrightarrow{\;k\;} & C' \\
 & & & & \bullet\downarrow\,{\scriptstyle\le} \\
 & & & & C
\end{array}$$

That is, the composition is given by $A \xrightarrow{[k|_* B']h} C'' \xrightarrow{\;\le\;} C$.

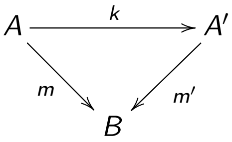
The functor **G**

The functor $\mathbf{G}\colon \mathbf{lcCat} \to \mathbf{oGpd}$ is defined as follows.

For a left cancellative category $\mathcal{C}$, the ordered groupoid $\mathbf{G}(\mathcal{C})$ has subobjects in $\mathcal{C}$ as objects; i.e., they are equivalence classes of arrows

$$m\colon A \to B$$

and $[m\colon A \to B] = [m'\colon A' \to B]$ if there is an isomorphism $k\colon A \xrightarrow{\sim} A'$ such that



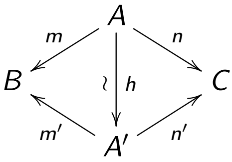
commutes.

The functor $\mathbf{G}$

The horizontal arrows in $\mathbf{G}(\mathcal{C})$ are equivalence classes of spans,

$$[m, n] \colon [m] \to [n]$$

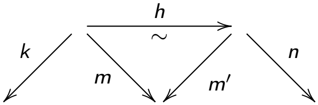
The equivalence relation is defined so that $[m, n] = [m', n']$ if and only if there is an isomorphism h making the following diagram commute:



The functor **G**

Composition of $[k, m]$ and $[m', n]$ is defined when $[m] = [m']$.

There is an isomorphism h such that $m'h = m$, giving rise to a diagram



in $\mathcal{C}$.

The composition is then $[k] \xrightarrow{[k,nh]} [nh] = [n]$.

The functor $\mathbf{G}$

The vertical arrows are given by

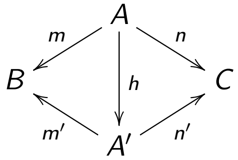
$$[n] \xrightarrow[\bullet]{\leq} [n']$$

if there is an arrow h in $\mathcal{C}$ such that $n = n'h$; i.e., $[n] \leq [n']$ as subobjects.

The functor $\mathbf{G}$

The order relation on horizontal arrows is defined by Lawson:

$[m, n] \leq [m', n']$ if there is an arrow h in $\mathcal{C}$ such that the diagram



commutes.

This implies then that $[m] \leq [m']$ and $[n] \leq [n']$.

The functor **G**

This partial order is what defines the double cells in $\mathbf{G}(\mathcal{C})$:

$$\begin{array}{ccc}
[m] & \xrightarrow{[m,n]} & [m] \\
\leq \big\downarrow^{\bullet} & \leq & {}^{\bullet}\big\downarrow \leq \\
[m'] & \xrightarrow[{[m',n']}]{} & [n']
\end{array}$$

Since there is at most one double cell for any frame of horizontal and vertical arrows, the horizontal and vertical composition of double cells is determined by the composition of the horizontal and vertical arrows.

The 2-equivalence

These constructions induce 2-functors, which establish the relationship between ordered groupoids and left cancellative categories.

Theorem

The 2-functors $\mathbf{L}\colon \mathbf{oGrpd} \to \mathbf{lcCat}$ *and* $\mathbf{G}\colon \mathbf{lcCat} \to \mathbf{oGrpd}$ *define a 2-adjoint equivalence of 2-categories,*

$$\mathbf{oGrpd} \simeq \mathbf{lcCat}.$$

Now, finally, what is an application of this equivalence?

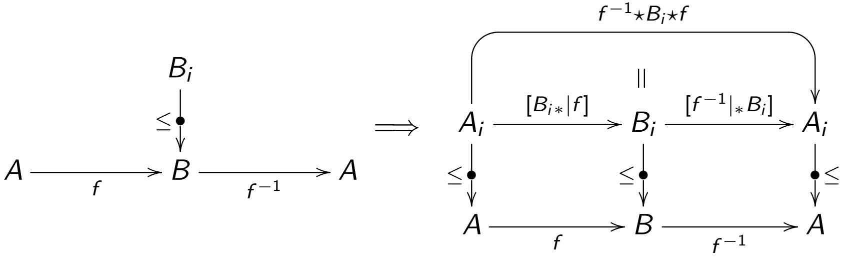
Lawson and Steinberg define Ehresmann topologies on ordered groupoids in terms of order (semi)ideals (i.e., down-closed subsets) of the principal order ideal

$$\downarrow A = \{a : a \leq A\}$$

of each object A.

These order ideals in an Ehresmann topology play the role of sieves in a Grothendieck topology, and an Ehresmann topology on an ordered groupoid $\mathcal{G}$ is then defined analogously as to how Grothendieck topologies are defined on an ordinary category, with pullback replaced with a certain $\star$-conjugation.

If $f : A \to B$ is a morphism of $\mathcal{G}$ with an object $B_i \leq B$, then the $\star$-conjugation of B_i by f is the composite of the appropriate restriction and corestriction of f^{-1} and f to B_i :



Like Lawson and Steinberg, we identify the partial identity $f^{-1} \star B_i \star f$ with its object A_i. We differ, however, in that our focus is less-so on the object A_i, and more-so on the fact that this A_i is the source of the corestriction $[B_{i*}|f]$.

Why the change in focus of A_i?

This sets us up to think of these easily in the context of double categories!

In particular:

- allows to think of Ehresmann topologies in the familiar terminology of sieves; since our vertical category is posetal, order ideals are exactly sieves of vertical arrows
- $\star$-conjugation is expressed as completion of double cells by corestricting horizontal arrows to elements of sieves on the codomain

Our Goal

Lawson and Steinberg then introduce Ehresmann topologies and give a correspondence between Ehresmann topologies on ordered groupoids and Grothendieck topologies on left cancellative categories, and prove that any etendue is equivalent to the category of sheaves on some Ehresmann site, by taking the an Ehresmann site coming from a left cancellative category.

They use the comparison lemma for left cancellative sites to get this result. We want to extend Lawson and Steinberg's work to include a comparison lemma for Ehresmann sites directly, and avoid having to move into the left cancellative categories.

This will leverage our adjoint equivalence between left cancellative categories and ordered groupoids as double categories.

Ehresmann topologies

We will compare Lawson and Steinberg's definition of an Ehresmann topology to our double categorical analogue.

Definition (Lawson/Steinberg)

Let $(\mathbf{G}, \circ, \leq)$ be an ordered groupoid. An *Ehresmann topology* on $\mathbf{G}$ is an assignment of, for each object $A \in \mathbf{G}$, a collection $T(A)$ of order ideals of $\downarrow A$ – called *covering ideals* – satisfying three axioms.

Definition (Double categories)

Let $\mathcal{G}$ be an ordered groupoid considered as a double category. An *Ehresmann topology* on $\mathcal{G}$ is an assignment of each object A of $\mathcal{G}$ to a set of vertical sieves $T(A)$ satisfying three axioms.

Axiom (ET1)

Lawson/Steinberg:

$\downarrow A \in T(A)$ for each object $A \in \mathbf{G}$.

Double categories:

The trivial sieve $\{A' \xrightarrow{\leq} A\}$ on A is in $T(A)$.

Axiom (ET2)

Lawson/Steinberg:

Let A and B be objects of **G** such that $B \leq_{\mathcal{J}} A$. Then for each $x : B \cong A' \leq A$ and $\mathcal{A} \in T(A)$, we have $x^{-1} \star \mathcal{A} \star x \in T(B)$.

Double categories:

For each horizontal arrow $f : A \to B$ and vertical covering sieve $\mathcal{B} = \{ B_i \overset{\leq}{\longrightarrow\!\!\!\!\!\bullet\!\!\!\!\to} B \} \in T(B)$, we have $\{A_i = \mathrm{hdom}[B_{i*}|f] : B_i \in \mathcal{B}\} \in T(A)$.

Axiom (ET3)

Lawson/Steinberg:

Let A be an object of $\mathbf{G}$, let $\mathcal{A} \in T(e)$ and let $\mathcal{B} \trianglelefteq \downarrow A$ be an arbitrary order ideal of $\downarrow A$. Suppose that, for each $x : B \cong A' \leq A$ (where $A' \in \mathcal{A}$), we have $x^{-1} \star \mathcal{B} \star x \in T(B)$. Then $\mathcal{B} \in T(A)$.

Double categories:

Let B be an object and let $\mathcal{B} = \{\, B_i \overset{\leq}{\longrightarrow\!\!\!\bullet\!\!\!\rightarrow} B \,\}$ be a vertical sieve on B. If, for each horizontal arrow $f : A \to B$, we have $\{A_i = \mathrm{hdom}[B_{i*}|f] : B_i \in \mathcal{B}\} \in T(A)$, then $\mathcal{B} \in T(B)$.

Which functors induce an equivalence of sheaf categories?

The Comparison Lemma (Kock/Moerdijk) gives five criteria that completely characterizes such functors for Grothendieck sites.

What are the analogous criteria for characterizing such functors for Ehresmann sites?

Criterion 1: Covering Preserving

In sites:

A functor $u : (\mathcal{C}, J) \to (\mathcal{C}', J')$ of sites is *covering preserving* means that, if $\xi \in J(C)$, then $u(\xi) \in J(u(C))$.

In Ehresmann sites:

A double functor $u : (\mathcal{G}, T) \to (\mathcal{G}', T')$ of Ehresmann sites is *covering preserving* means that, if $\mathcal{A} \in T(A)$, then $u(\mathcal{A}) \in T(u(A))$.

Criterion 2: Locally Full

In sites:

A functor $u : (\mathcal{C}, J) \to (\mathcal{C}', J')$ of sites is *locally full* means that, if $g : u(C) \to u(D)$ is an arrow in $\mathcal{C}'$, there exists a cover $(\xi_i : C_i \to C)_{i \in I}$ in $\mathcal{C}$ and maps $(f_i : C_i \to D)_{i \in I}$ such that $g \circ u(\xi_i) = u(f_i)$ for all $i \in I$.

In Ehresmann sites:

A double functor $u : (\mathcal{G}, T) \to (\mathcal{G}', T')$ of Ehresmann sites is *locally full* means that, if $g' : u(A) \to u(B)$ is a horizontal arrow in $\mathcal{G}'$, then there exists a covering vertical sieve $\{A_i\}_{i \in I} \in T(A)$ and a family of horizontal arrows $\{f_i : A_i \to B_i\}_{i \in I}$ in $\mathcal{G}$ such that $[f'|_* u(A_i)] = u(f_i)$ for all $i \in I$.

Criterion 3: Locally Faithful

In sites:

A functor $u : (\mathcal{C}, J) \to (\mathcal{C}', J')$ of sites is *locally faithful* means that, if $f, f' : C \to D$ in $\mathcal{C}$ with $u(f) = u(f')$, then there exists a cover $(\xi_i)_{i \in I}$ of C with $f \circ \xi_i = f' \circ \xi_i$ for all $i \in I$.

In Ehresmann sites:

A double functor $u : (\mathcal{G}, T) \to (\mathcal{G}', T')$ of Ehresmann sites is *locally faithful* means that, if $f, g : A \to B$ are horizontal arrows in $\mathcal{G}$ with $u(f) = u(g)$, then there exists a covering vertical sieve $\{A_i\}_{i \in I} \in T(A)$ with $[f|_* A_i] = [g|_* A_i]$ for all $i \in I$.

Criterion 4: Locally Surjective on Objects

In sites:

A functor $u : (\mathcal{C}, J) \to (\mathcal{C}', J')$ of sites is *locally surjective on objects* means that, for each object C' of $\mathcal{C}'$, there exists a covering family of the form $(u(C_i) \to C')_{i \in I}$.

In Ehresmann sites:

A double functor $u : (\mathcal{G}, T) \to (\mathcal{G}', T')$ of Ehresmann sites is *locally surjective on objects* means that, for each object A' of $\mathcal{G}'$, there is a set $\{A_i\}_{i \in I}$ of objects in $\mathcal{G}$ such that $\{u(A_i)\}_{i \in I} \in T'(A')$.

Criterion 5: Co-continuous

In sites:

A functor $u : (\mathcal{C}, J) \to (\mathcal{C}', J')$ of sites is *co-continuous* means that if $(\xi_i : C'_i \to u(C))_{i \in I}$ is a cover in $\mathcal{C}'$, then the set of arrows $f : D \to C$ in $\mathcal{C}$, such that $u(f)$ factors through some ξ_i, covers C in $\mathcal{C}$.

In Ehresmann sites:

A double functor $u : (\mathcal{G}, T) \to (\mathcal{G}', T')$ of Ehresmann sites is *co-continuous* means that if $\{A'_i\}_{i \in I} \in T(u(A))$ is a covering vertical sieve of $u(A)$ in $\mathcal{G}'$, then the set $\{A_j \leq A : u(A_j) \leq A'_i \text{ for some } i \in I\}$ is a covering vertical sieve of A in $\mathcal{G}$.

Theorem (Comparison Lemma for Sites)

Let $u : (\mathcal{C}, J) \to (\mathcal{C}', J')$ be a functor of essentially small sites. If u satisfies (1) – (4), then if F is a sheaf on $\mathcal{G}'$, then Fu is a sheaf on $\mathcal{G}$ (i.e., u is continuous), and the functor $u^* : \mathrm{sh}(\mathcal{C}', J') \to \mathrm{sh}(\mathcal{C}, J)$ is full and faithful. If further u satisfies (5), then u^* is an equivalence. □

Given the criteria expressed in the language of Ehresmann sites, the statement of a new comparison lemma is straight forward.

Theorem (Comparison Lemma for Ehresmann Sites)

Let $u : (\mathcal{G}, T) \to (\mathcal{G}', T')$ be a functor of Ehresmann sites. If u satisfies (1) – (4), then if F is a sheaf on $\mathcal{G}'$, then Fu is a sheaf on $\mathcal{G}$ (i.e., u is continuous), and the functor $u^* : \mathrm{sh}(\mathcal{G}', T') \to \mathrm{sh}(\mathcal{G}, T)$ is full and faithful. If further u satisfies (5), then u^* is an equivalence. □