

# Restriction monads and algebras.

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# Monads for Partial Computation

Moggi (1991) introduces monads as abstract notions of computation.

In particular, the monad

$$T = \text{Par} : \mathbf{Set} \rightarrow \mathbf{Set} : A \mapsto A \amalg \{*\}$$

can be used to re-interpret a partial function of sets  $f : A \rightarrow B$  as a total function

$$f_T : A \rightarrow \text{Par}(B)$$

where

$$f_T(a) = \begin{cases} f(a) & \text{if } f(a) \text{ is defined.} \\ * & \text{if } f(a) \text{ is not defined.} \end{cases}$$

Then composing partial functions can be done by composing their corresponding Kleisli arrows.

# Monads for Partial Computation

Can we use monads to encode partial computation in some category other than set?

With what additional structure must we equip a monad

$$T : \mathbf{X} \rightarrow \mathbf{X}$$

to encode, in some sense, that this monad is partially defined without requiring any additional structure on the category  $\mathbf{X}$  itself?

We will call such a thing a *restriction monad*.

We need to choose that sense in which a monad is partially defined.

# Restriction Categories

Restriction monads will model partial computation in the same way that restriction categories (Cockett and Lack) model partial maps.

A category  $\mathbf{X}$  is called a restriction category when it can be equipped with an assignment

$$(f : A \rightarrow B) \mapsto (\bar{f}_A : A \rightarrow A)$$

of all arrows  $f$  in  $\mathbf{X}$  to an endomorphism  $\bar{f}$  satisfying:

1. For all maps  $f$ ,  $f \bar{f}_A = f$ .
2. For all maps  $f : A \rightarrow B$  and  $g : A \rightarrow B'$ ,  $\bar{f}_A \bar{g}_A = \bar{g}_A \bar{f}_A$ .
3. For all maps  $f : A \rightarrow B$  and  $g : A \rightarrow B'$ ,  $\overline{g_A f_A} = \bar{g}_A \bar{f}_A$ .
4. For all maps  $f : B \rightarrow A$  and  $g : A \rightarrow B'$ ,  $\bar{g}_A f = f \overline{(gf)_B}$ .

# Restriction Monads

We will define a restriction monad in a bicategory.

We need a 0-cell  $x$  with a 1-cell  $T : x \rightarrow x$ .

With restriction categories in mind, what additional data do we need in a restriction monad?

- ▶ We need to assign special “endomorphisms”, so we need some data to hold these. We use another 1-cell  $E : x \rightarrow x$  for this.
- ▶ We need to be able to pick out the “source” to anchor these endomorphisms. We use a 1-cell  $D : x \rightarrow x$ .

On these data, we need a “restriction operator”: a 2-cell

$$\rho : D \Longrightarrow E$$

with some other 2-cells so that we can express some suitable axioms reminiscent of restriction categories.

# Restriction Monads in $\text{Span}(\mathbf{Set})$

Let's define these 2-cells, and look at the restriction axioms in the context of an example.

If  $\mathbf{X}$  is a restriction category, consider the (ordinary) monad  $R(\mathbf{X})$  in  $\text{Span}(\mathbf{Set})$  corresponding to its underlying category.

$$T = \begin{array}{ccc} & \mathbf{X}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

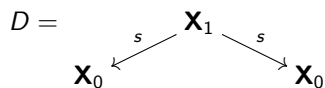
$$\eta : 1_T \Rightarrow T : \mathbf{X}_0 \rightarrow \mathbf{X}_1 : A \mapsto 1_A$$

$$\begin{array}{ccccc} & & \mathbf{X}_0 & & \\ & 1 \swarrow & & \searrow 1 & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & \downarrow \eta & \searrow t & \\ & & \mathbf{X}_1 & & \end{array}$$

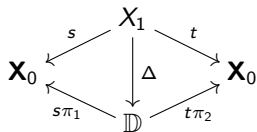
$$\mu : T^2 \Rightarrow T : \mathbb{C} \rightarrow \mathbf{X}_1 : (f, g) \mapsto gf$$

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & s\pi_1 \swarrow & & \searrow t\pi_2 & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & \downarrow \mu & \searrow t & \\ & & \mathbf{X}_1 & & \end{array}$$

Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$



$$\Delta : T \Rightarrow TD : \mathbf{X}_1 \rightarrow \mathbb{D} : f \mapsto (f, f)$$



$$\mathbb{D} = \{(f, g) \in \mathbf{X}_1 \times \mathbf{X}_1 : sf = sg\}$$

Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

$$E = \begin{array}{ccc} & \bar{\mathbf{X}}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

where

$$\bar{\mathbf{X}}_1 = \{\bar{f} : f \in \mathbf{X}\}$$

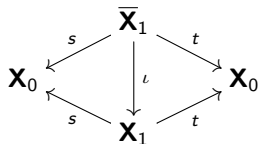
And define

$$\rho : D \Rightarrow E : \mathbf{X}_1 \rightarrow \bar{\mathbf{X}}_1 : f \mapsto \bar{f}$$

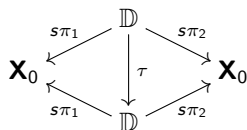


Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

$$\iota : E \Rightarrow T : \bar{\mathbf{X}}_1 \rightarrow \mathbf{X}_1 : \bar{f} \mapsto \bar{f}$$

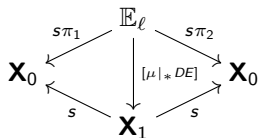


$$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \rightarrow \mathbb{D} : (f, g) \mapsto (g, f)$$

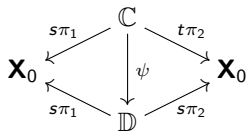


Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

$$[\mu |_* DE] : DE \Rightarrow D : \mathbb{E}_\ell \rightarrow \mathbf{X}_1 : (\bar{f}, g) \mapsto g\bar{f}$$



$$\psi : DT \Rightarrow TD : \mathbb{C} \rightarrow \mathbb{D} : (f, g) \mapsto (gf, f)$$



Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

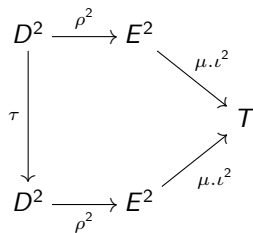
(R.1): " $f = f\bar{f}$ "

$$\begin{array}{ccc}
 T & \xrightarrow{\Delta} & TD \\
 \parallel & & \downarrow T\rho \\
 1_T & & TE \\
 \parallel & & \leftarrow \mu \cdot T\iota \\
 T & & 
 \end{array}$$

$$f \mapsto (f, f) \mapsto (\bar{f}, f) \mapsto f\bar{f}$$

Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

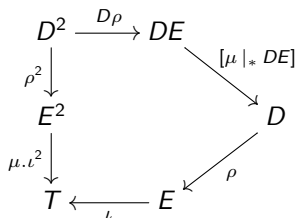
(R.2): " $\bar{f}\bar{g} = \bar{g}\bar{f}$ "



$$(f, g) \mapsto (\bar{f}, \bar{g}) \mapsto \bar{g}\bar{f}$$

Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

(R.3): " $\overline{g\bar{f}} = \bar{g}\bar{f}$ "



$$(f, g) \mapsto (\bar{f}, g) \mapsto g\bar{f} \mapsto \overline{g\bar{f}}$$

Example:  $R(\mathbf{X}) : \mathbf{X}_0 \rightarrow \mathbf{X}_0$  in  $\text{Span}(\mathbf{Set})$

(R.4): " $\overline{gf} = f\overline{gf}$ "

$$\begin{array}{ccccc}
 DT & \xrightarrow{\rho T} & ET & \xrightarrow{\mu \cdot \iota T} & T \\
 \downarrow \psi & & & & \uparrow \mu \cdot T \iota \\
 TD & \xrightarrow{T\rho} & & & TE
 \end{array}$$

$$(f, g) \mapsto (gf, f) \mapsto (\overline{gf}, f) \mapsto f\overline{gf}$$

## From $\text{Span}(\mathbf{Set})$ to $\mathbf{rCat}$

In the ordinary case, there is a bijective correspondence between small categories and monads in  $\text{Span}(\mathbf{Set})$ .

In the case of small restriction categories and restriction monads in  $\text{Span}(\mathbf{Set})$ , this is weakened to an adjunction<sup>1</sup>.

This corresponds to a restriction operator being structure rather than a property; there are many ways to equip the same category with distinct restriction structures.

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<sup>1</sup>Not in bicategories, though, need to consider these as monads in double categories à la Fiore et al.

# Algebras for Restriction Monads

A right algebra  $(S, h)$  for a restriction monad consists of a 1-cell  $S : x \rightarrow y$  together with 2-cells

- ▶  $h^T : ST \Rightarrow S$ ,
- ▶  $h^D : SD \Rightarrow S$ ,
- ▶  $h^E : SE \Rightarrow S$  and
- ▶  $r : D \Rightarrow SD$ .

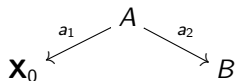
This time,  $r : S \Rightarrow SD$  plays the role of restriction operator when post-composed with  $S\rho$ .

As well as the usual associative and unit laws, we have a host of restriction-category-flavoured conditions which are again best understood in the context of an example.

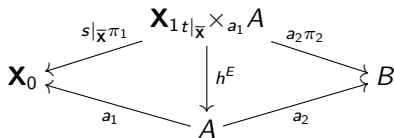
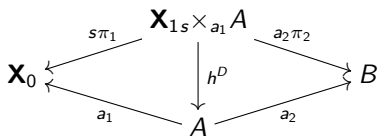
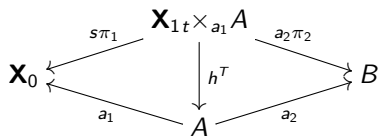


# Algebras for $R(\mathbf{X})$

In  $\text{Span}(\mathbf{Set})$ , such an algebra is a span



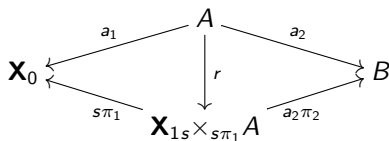
with span morphisms



## Algebras for $R(\mathbf{X})$

Recall that  $h^T$  is simply the action of  $\mathbf{X}_1$  on  $A$ , or that algebras for  $T$  as an ordinary monad in  $\text{Span}(\mathbf{Set})$  is a module.

The span morphism



Can then be post-composed with  $S\rho$  :

$$a \xrightarrow{r} (f_a, ra) \xrightarrow{S\rho} (\overline{f_a}, ra)$$

So to each element of  $A$  can be associated a restriction idempotent  $\overline{f_a}$ . This assignment gives the module the structure of a *restriction module*.

# Restriction Modules

A restriction (left  $\mathbf{Y}$ -, right  $\mathbf{X}$ -bi)module  $\varphi : \mathbf{X} \dashv \! \dashv \! \rightarrow \mathbf{Y}$  is a collection

$$\{\varphi(y, x) : y \in \mathbf{Y}_0, x \in \mathbf{X}_0\}$$

of sets indexed by the objects of  $\mathbf{X}$  and  $\mathbf{Y}$  together with:

- ▶ for all objects  $y, y' \in \mathbf{Y}$  and  $x, x' \in \mathbf{X}$ , a pair of action maps

$$\lambda_{y', y, x}^\varphi : \mathbf{Y}(y', y) \times \varphi(y, x) \longrightarrow \varphi(y', x)$$

$$\rho_{y, x, x'}^\varphi : \varphi(y, x) \times \mathbf{X}(x, x') \longrightarrow \varphi(y, x')$$

We will write both  $\lambda(g, \alpha)$  and  $\rho(\alpha, f)$  using the dot notation  $g \cdot \alpha$  and  $\alpha \cdot f$ .

- ▶ a map assigning each  $\alpha \in \varphi(y, x)$  to some  $\bar{\alpha} : x \rightarrow x$  in  $\mathbf{X}$  satisfying some conditions (next slide).

## Restriction Modules

Again, the conditions that this assignment of each  $\alpha \in \varphi(y, x)$  to some  $\bar{\alpha} : x \rightarrow x$  in  $\mathbf{X}$  should not be too surprising.

1. for each  $\alpha \in \varphi(y, x)$ ,  $\bar{\alpha} = \bar{f}$  for some  $f : x \rightarrow x'$  in  $\mathbf{X}$ ;
2. for each  $\alpha \in \varphi(y, x)$ ,  $\alpha \cdot \bar{\alpha} = \alpha$ ;
3. for each  $\alpha \in \varphi(y, x)$  and  $\beta \in \varphi(y', x)$ ,  $\bar{\alpha} \circ \bar{\beta} = \bar{\beta} \circ \bar{\alpha}$ ;
4. for each  $\alpha \in \varphi(y, x)$  and  $\beta \in \varphi(y', x)$ ,  $\overline{\alpha \cdot \beta} = \bar{\alpha} \circ \bar{\beta}$ ;
5. (a) for each  $\alpha \in \varphi(y, x)$  and  $f : x' \rightarrow x$  in  $\mathbf{X}$ ,  $\bar{\alpha} \circ f = f \circ \overline{\alpha \cdot f}$ ;  
(b) for each  $\alpha \in \varphi(y, x)$  and  $g : y \rightarrow y'$  in  $\mathbf{Y}$ ,  $\bar{g} \cdot \alpha = \alpha \cdot \overline{g \cdot \alpha}$ .

# Double Categories

We can organize these data into two double categories, related by adjunction.

	$\mathbf{rMon}(\mathbf{rCat})$	$\mathbf{rMod}(\mathbf{Span}(\mathbf{Set}))$
Objects	Rest. Cats.	Rest. Monads in $\mathbf{Span}(\mathbf{Set})$
Vertical Arrows	Rest. Functors	Monad Morphisms
Horizontal Arrows	Rest. Modules	Algebras
Double Cells	Equivariant Maps	Equivariant Maps

$$\begin{array}{ccc}
 \mathbf{X} & \xrightarrow{M} & \mathbf{X}' \\
 F \downarrow & \alpha & \downarrow F' \\
 \mathbf{Y} & \xrightarrow{M'} & \mathbf{Y}'
 \end{array}$$

$\mathbf{rMon}(\mathbf{rCat})$

$$\begin{array}{ccc}
 T & \xrightarrow{A} & T' \\
 F \downarrow & \alpha & \downarrow F' \\
 N & \xrightarrow{B} & N'
 \end{array}$$

$\mathbf{rMod}(\mathbf{Span}(\mathbf{Set}))$

And these are *double restriction (bi)categories* in the sense that we can assign to each module  $M$  some  $\bar{M}$  which behaves as the restriction idempotent of  $M$ .