

Restriction Monads and Category Objects

FMCS at UBC

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Double Restriction Categories

Interested in double restriction categories.

- Double cells, etc. with restriction structure in both directions.
How should these behave?
- Would be helpful then to have some notion of a restriction category internal to the category of restriction categories.

Restriction Categories

A category \mathbf{X} is called a restriction category when it can be equipped with an assignment

$$(f : A \rightarrow B) \mapsto (\overline{f}_A : A \rightarrow A)$$

of all arrows f in \mathbf{X} to an endomorphism \overline{f} satisfying:

- 1 For all maps f , $f \overline{f}_A = f$.
- 2 For all maps $f : A \rightarrow B$ and $g : A \rightarrow B'$, $\overline{f}_A \overline{g}_A = \overline{g}_A \overline{f}_A$.
- 3 For all maps $f : A \rightarrow B$ and $g : A \rightarrow B'$, $\overline{g}_A \overline{f}_A = \overline{g}_A \overline{f}_A$.
- 4 For all maps $f : B \rightarrow A$ and $g : A \rightarrow B'$, $\overline{g}_A f = f (\overline{g f})_B$.

Restriction Category Objects

- Obvious data needed: object of arrows, objects, composition map, pullbacks, source and target, unit and restriction map $\rho : \mathbf{X}_1 \rightarrow \mathbf{X}_0$
- More data needed to allow us to diagrammatically express (R.1) - (R.4).
- Also, want to keep an eye out and avoid using the fact that restriction categories are internal to **Set**

Restriction Monads: the approach

Recall that monads in $\mathbf{Span}(\mathbf{Set})$ are small categories. We will define restriction monads in a way that restriction monads in $\mathbf{Span}(\mathbf{Set})$ are small restriction categories.

Let \mathbf{X} be a restriction category and let us define a restriction monad $R(\mathbf{X})$ in $\mathbf{Span}(\mathbf{Set})$.

Restriction Monads: the construction $R(\mathbf{X})$

The necessary data:

$$T = \begin{array}{ccc} & \mathbf{X}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

$$\eta : 1_T \Rightarrow T : \mathbf{X}_0 \rightarrow \mathbf{X}_1 : A \mapsto 1_A$$

$$\begin{array}{ccccc} & & \mathbf{X}_0 & & \\ & 1 \swarrow & & \searrow 1 & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & \eta \downarrow & \searrow t & \\ & & \mathbf{X}_1 & & \end{array}$$

$$\mu : T^2 \Rightarrow T : \mathbb{C} \rightarrow \mathbf{X}_1 : (f, g) \mapsto gf$$

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & s\pi_1 \swarrow & & \searrow t\pi_2 & \\ \mathbf{X}_0 & & & & \mathbf{X}_0 \\ & s \swarrow & \mu \downarrow & \searrow t & \\ & & \mathbf{X}_1 & & \end{array}$$

Restriction Monads: the construction $R(\mathbf{X})$

Need to encode the restriction operator. A naive choice could be a morphism of spans

$$\rho : T \Rightarrow T$$

This implies that

$$sf = s(\rho f) \text{ and } tf = t(\rho f)$$

Restriction Monads: the construction $R(\mathbf{X})$

Attempting to fix this, define instead

$$\rho : D \Rightarrow T$$

where

$$D = \begin{array}{ccc} & \mathbf{X}_1 & \\ s \swarrow & & \searrow s \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

This implies that

$$sf = s(\rho f) \text{ and } sf = t(\rho f)$$

Restriction Monads: the construction $R(\mathbf{X})$

Using this new 2-cell we can express (R.1) diagrammatically by imposing that the following diagram commutes:

$$\begin{array}{ccc}
 T & \xrightarrow{\Delta} & TD \\
 & \swarrow \mu & \downarrow T\rho \\
 & & T^2
 \end{array}$$

where

$$\Delta : T \Rightarrow TD : \mathbf{X}_1 \rightarrow \mathbb{D} : f \mapsto (f, f)$$

$$\begin{array}{ccccc}
 & & \mathbf{X}_1 & & \\
 & s \swarrow & \downarrow \Delta & \searrow t & \\
 \mathbf{X}_0 & & \mathbb{D} & & \mathbf{X}_0 \\
 & \swarrow s\pi_1 & & \searrow t\pi_2 & \\
 & & & &
 \end{array}$$

$$\mathbb{D} = \{(f, g) \in \mathbf{X}_1 \times \mathbf{X}_1 : sf = sg\}$$

Restriction Monads: the construction $R(\mathbf{X})$

Consider encoding (R.3): $\overline{g\bar{f}} = \bar{g}\bar{f}$, where $\text{dom}f = \text{dom}g$. There are two places to “start”:

- D^2 :

$$D^2 \xrightarrow{D\rho} DT$$

$$(f, g) \longmapsto (\bar{f}, g)$$

- TD :

$$TD \xrightarrow{T\rho} T^2$$

Both ways: you get stuck.

Restriction Monads: the construction $R(\mathbf{X})$

Solution: “weaken” when we are able to compose. Idea:
 We know (in $\mathbf{Span}(\mathbf{Set})$) that the map

$$D^2 \xrightarrow{D\rho} DT$$

$$(f, g) \longmapsto (\bar{f}, g)$$

gives what should be a composable pair. Would like to keep track
 of these morally, yet not typely, composable pairs.

Restriction Monads: the construction $R(\mathbf{X})$

We don't need all of the composites. Turns out we can fix this by keeping track only of composites involving restriction idempotents. This can be done with new data:

$$E = \begin{array}{ccc} & \bar{\mathbf{X}}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

where

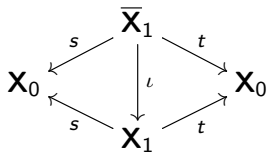
$$\bar{\mathbf{X}}_1 = \{\bar{f} : f \in \mathbf{X}\}$$

And redefine

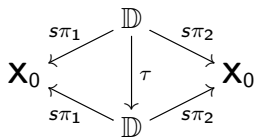
$$\rho : D \Rightarrow E$$

Restriction Monads: the construction $R(\mathbf{X})$

$$\iota : E \Rightarrow T : \bar{\mathbf{X}}_1 \rightarrow \mathbf{X}_1 : \bar{f} \mapsto \bar{f}$$

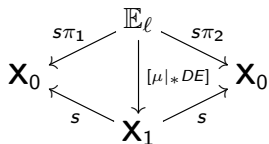


$$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \rightarrow \mathbb{D} : (f, g) \mapsto (g, f)$$

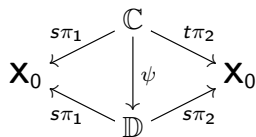


Restriction Monads: the construction $R(\mathbf{X})$

$$[\mu |_* DE] : DE \Rightarrow D : \mathbb{E}_\ell \rightarrow \mathbf{X}_1 : (\bar{f}, g) \mapsto g\bar{f}$$



$$\psi : DT \Rightarrow TD : \mathbb{C} \rightarrow \mathbb{D} : (f, g) \mapsto (gf, f)$$



Restriction Monads: the construction $R(\mathbf{X})$

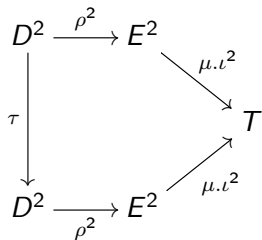
(R.1): $f = f\bar{f}$

$$\begin{array}{ccc}
 T & \xrightarrow{\Delta} & TD \\
 \parallel^{1_T} & & \downarrow T\rho \\
 T & \xleftarrow{\mu \cdot T\nu} & TE
 \end{array}$$

$$f \mapsto (f, f) \mapsto (\bar{f}, f) \mapsto f\bar{f}$$

Restriction Monads: the construction $R(\mathbf{X})$

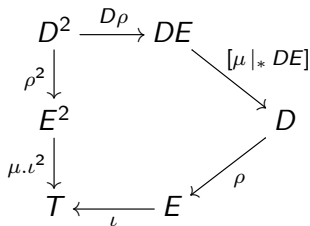
(R.2): $\bar{f} \bar{g} = \bar{g} \bar{f}$



$$(f, g) \mapsto (\bar{f}, \bar{g}) \mapsto \bar{g} \bar{f}$$

Restriction Monads: the construction $R(\mathbf{X})$

(R.3): $\overline{g\bar{f}} = \bar{g}\bar{f}$



$$(f, g) \mapsto (\bar{f}, g) \mapsto g\bar{f} \mapsto \overline{g\bar{f}}$$

Restriction Monads: the construction $R(\mathbf{X})$

(R.4):

$$\begin{array}{ccccc}
 DT & \xrightarrow{\rho T} & ET & \xrightarrow{\mu \cdot \iota T} & T \\
 \downarrow \psi & & & & \uparrow \mu \cdot T \iota \\
 TD & \xrightarrow{T \rho} & & & TE
 \end{array}$$

$$(f, g) \mapsto (gf, f) \mapsto (\overline{gf}, f) \mapsto f \overline{gf}$$

Restriction Monads: a definition

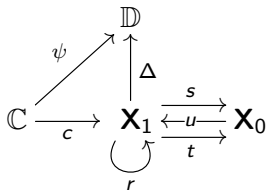
In a bicategory with involution, a restriction monad consists of a 0-cell x , 1-cells $T, D, E : x \rightarrow x$ and 2-cells

- $\eta : 1_T \Rightarrow T$,
- $\mu : T^2 \Rightarrow T, [\mu|_* DE] : DE \Rightarrow D$,
- $\rho : D \Rightarrow E$ (epic),
- $\iota : E \Rightarrow T$ (monic),
- $\Delta : T \Rightarrow TD, \tau : D^2 \Rightarrow D^2$ and
- $\psi : DT \Rightarrow TD$

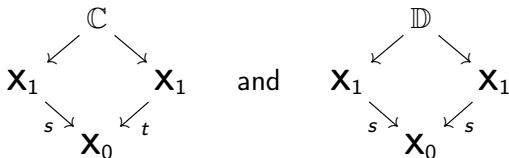
satisfying conditions (R.1) through (R.4) plus the usual monad laws plus $D^*D = DD^*$

Restriction Category Objects

A restriction category (in **Set**) contains the following data:



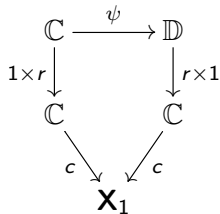
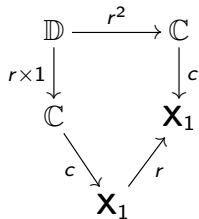
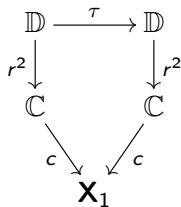
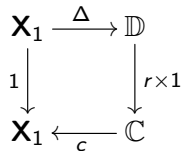
where \mathbb{C} and \mathbb{D} are defined by the pullback squares



Restriction Category Objects

- (i) $sr = s = tr$,
- (ii) $c = \pi_1\psi$ and $\pi_1 = \pi_2\psi$,
- (iii) associativity and unit laws from categories,
- (iv) $\pi_1\Delta = 1 = \pi_2\Delta$,
- (v) $\pi_1 = \pi_2\tau$ and $\pi_2 = \pi_1\tau$,

Restriction Category Objects



Restriction Category Objects

Definition

A double restriction category is a restriction category internal to \mathbf{rCat} .

Thank you.