Restriction Monads and Category Objects
FMCS at UBC

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Interested in double restriction categories.

- Double cells, etc. with restriction structure in both directions. How should these behave?
- Would be helpful then to have some notion of a restriction category internal to the category of restriction categories.
A category $\mathbf{X}$ is called a restriction category when it can be equipped with an assignment

$$\left( f : A \to B \right) \mapsto \left( f_A : A \to A \right)$$

of all arrows $f$ in $\mathbf{X}$ to an endomorphism $\overline{f}$ satisfying:

1. For all maps $f$, $f \overline{f_A} = f$.
2. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f_A g_A} = \overline{g_A f_A}$.
3. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g_A f_A} = \overline{g_A f_A}$.
4. For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g_A} f = f (gf)_B$. 

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Restriction Monads and Category Objects
Obvious data needed: object of arrows, objects, composition map, pullbacks, source and target, unit and restriction map $\rho : X_1 \to X_0$.

More data needed to allow us to diagrammatically express (R.1) - (R.4).

Also, want to keep an eye out and avoid using the fact that restriction categories are internal to $\textbf{Set}$. 
Recall that monads in $\text{Span}(\text{Set})$ are small categories. We will define restriction monads in a way that restriction monads in $\text{Span}(\text{Set})$ are small restriction categories.

Let $\mathbf{X}$ be a restriction category and let us define a restriction monad $R(\mathbf{X})$ in $\text{Span}(\text{Set})$. 
Restriction Monads: the construction $R(X)$

The necessary data:

$$T = X_0 \xrightarrow{s} X_1 \xleftarrow{t} X_0$$

$$\eta: 1_T \Rightarrow T: X_0 \rightarrow X_1 : A \mapsto 1_A$$

$$\mu: T^2 \Rightarrow T: C \rightarrow X_1 : (f, g) \mapsto gf$$
Need to encode the restriction operator. A naive choice could be a morphism of spans

\[ \rho : T \Rightarrow T \]

This implies that

\[ sf = s(\rho f) \quad \text{and} \quad tf = t(\rho f) \]
Restriction Monads: the construction $R(X)$

Attempting to fix this, define instead

$$\rho : D \Rightarrow T$$

where

$$D = \xymatrix{ & X_1 \ar[l]_s 
& \ar[r]^s \ar[l]_s \ar[r] \ar[l] & X_0 \ar[l]_s}

This implies that

$$sf = s(\rho f) \text{ and } sf = t(\rho f)$$
Restriction Monads: the construction $R(X)$

Using this new 2-cell we can express (R.1) diagrammatically by imposes that the following diagram commutes:

$$
\begin{array}{ccc}
T & \xrightarrow{\Delta} & TD \\
\downarrow{\mu} & & \downarrow{T\rho} \\
T^2 & & \\
\end{array}
$$

where

$$\Delta : T \Rightarrow TD : X_1 \rightarrow D : f \mapsto (f, f)$$

$$D = \{(f, g) \in X_1 \times X_1 : sf = sg\}$$
Consider encoding (R.3): \( \overline{g \bar{f}} = \overline{g} \bar{f} \), where \( \text{dom} f = \text{dom} g \). There are two places to “start”:

- **\( D^2 \):**
  \[
  D^2 \xrightarrow{D\rho} DT
  \]
  \[
  (f, g) \xleftarrow{} (\bar{f}, g)
  \]

- **\( TD \):**
  \[
  TD \xrightarrow{T\rho} T^2
  \]
  Both ways: you get stuck.
Solution: “weaken” when we are able to compose. Idea:
We know (in $\text{Span}(\text{Set})$) that the map

$$D^2 \xrightarrow{D\rho} DT$$

$$\quad (f, g) \quad \mapsto \quad (\overline{f}, g)$$

gives what should be a composable pair. Would like to keep track of these morally, yet not typely, composable pairs.
We don’t need all of the composites. Turns out we can fix this by keeping track only of composites involving restriction idempotents. This can be done with new data:

\[ E = \begin{align*}
X_0 & \xrightarrow{s} \overline{X}_1 \\
\overline{X}_1 & \xleftarrow{t} X_0
\end{align*} \]

where

\[ \overline{X}_1 = \{ \overline{f} : f \in X \} \]

And redefine

\[ \rho : D \Rightarrow E \]
Restriction Monads: the construction $R(X)$

$$\nu : E \Rightarrow T : \overline{X_1} \rightarrow X_1 : f \mapsto \overline{f}$$

$$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \rightarrow \mathbb{D} : (f, g) \mapsto (g, f)$$
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Restriction Monads: the construction $R(X)$

$$[\mu \mid \ast \ DE] : DE \Rightarrow D : E_{\ell} \rightarrow X_1 : (\bar{f}, g) \mapsto g\bar{f}$$

$$\psi : DT \Rightarrow TD : C \rightarrow D : (f, g) \mapsto (gf, f)$$

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Restriction Monads: the construction $R(X)$

(R.1): $f = f f$

\[
\begin{array}{ccc}
T & \xrightarrow{\Delta} & TD \\
\downarrow{1_T} & & \downarrow{T \rho} \\
T & \xleftarrow{\mu.T \iota} & TE \\
\end{array}
\]

$f \mapsto (f, f) \mapsto (f, f) \mapsto f f$
(R.2): $\bar{f} \bar{g} = \bar{g} \bar{f}$

\[ (f, g) \mapsto (\bar{f}, \bar{g}) \mapsto \bar{g} \bar{f} \]
(R.3): $\overline{gf} = \overline{g} \overline{f}$

$D^2 \xrightarrow{D\rho} DE \xrightarrow{[\mu \ast DE]} D$

$\rho^2 \downarrow \quad \downarrow \rho^2$

$E^2 \xrightarrow{\mu \cdot \iota^2} T \xleftarrow{\iota} E$

$(f, g) \mapsto (\overline{f}, \overline{g}) \mapsto g\overline{f} \mapsto \overline{gf}$
(R.4):

\[
\begin{align*}
DT & \xrightarrow{\rho^T} ET & \xrightarrow{\mu.T} T \\
TD & \xrightarrow{T\rho} TE
\end{align*}
\]

\[
(f, g) \mapsto (gf, f) \mapsto (gf, f) \mapsto f \overline{gf}
\]
In a bicategory with involution, a restriction monad consists of a 0-cell \( x \), 1-cells \( T, D, E : x \to x \) and 2-cells

- \( \eta : 1_T \Rightarrow T \),
- \( \mu : T^2 \Rightarrow T \), \([\mu|_* DE] : DE \Rightarrow D\),
- \( \rho : D \Rightarrow E \) (epic),
- \( \iota : E \Rightarrow T \) (monic),
- \( \Delta : T \Rightarrow TD \), \( \tau : D^2 \Rightarrow D^2 \) and
- \( \psi : DT \Rightarrow TD \)

satisfying conditions (R.1) through (R.4) plus the usual monad laws plus \( D^* D = DD^* \)
A restriction category (in $\textbf{Set}$) contains the following data:

\[
\begin{array}{c}
\mathbb{C} \\
c \downarrow \\
\downarrow \psi \\
\mathbb{D} \\
\downarrow \Delta \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\mathbb{X}_1 \\
\downarrow s \\
\downarrow t \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\mathbb{X}_0 \\
\hline
\end{array}
\]

where $\mathbb{C}$ and $\mathbb{D}$ are defined by the pullback squares

\[
\begin{array}{c}
\mathbb{X}_1 \\
\downarrow s \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\mathbb{X}_0 \\
\hline
\end{array}
\]

and

\[
\begin{array}{c}
\mathbb{X}_1 \\
\downarrow s \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\downarrow \downarrow \\
\mathbb{X}_0 \\
\hline
\end{array}
\]
Restriction Category Objects

(i) $sr = s = tr$,  
(ii) $c = \pi_1 \psi$ and $\pi_1 = \pi_2 \psi$,  
(iii) associativity and unit laws from categories,  
(iv) $\pi_1 \Delta = 1 = \pi_2 \Delta$,  
(v) $\pi_1 = \pi_2 \tau$ and $\pi_2 = \pi_1 \tau$,  

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Definition

A double restriction category is a restriction category internal to $r\text{Cat}$. 
Thank you.