

Inverse Restriction Categories and Their Groupoids

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higher-dimensional algebra:

- Ronald Brown: 2-dimensional group theory
- Kock: Double (inverse) semigroups (2007)
- Edmunds: Double magma (2013) and interchange rings (2014)
- Bremner and Madariaga : double semigroups (2014)

Double inverse semigroups: Element-based definition

Definition (Kock, 2007)

A *double inverse semigroup* (S, \odot, \circ) is a set S such that (S, \odot) and (S, \circ) are inverse semigroups and \odot and \circ satisfy the middle-four interchange law:

$$(a \circ b) \odot (c \circ d) = (a \odot c) \circ (b \odot d)$$

Eckmann-Hilton-like result:

Theorem

Double inverse semigroups are improper [DeWolf and Pronk] and commutative [Kock].

Double inverse semigroups: Brown-style definition

Ronald Brown: double groups are single object double groupoids

Definition

A *double inverse semigroup* is a single object double "inverse semigroupoid".

"inverse semigroupoid" is the multiobject version of inverse semigroup

What are these and what do we know about them?

First question: what is an "inverse semigroupoid"?

At each object, we want an inverse semigroup:

- pseudoinverses ($xyx = x$ and $yxy = y$)
- commuting idempotents

What satisfies this need:

inverse restriction categories

Definition (Cockett and Lack, 2002)

A restriction structure on a category \mathbf{X} is an assignment of an arrow $\overline{f}_A : A \rightarrow A$ to each arrow $f : A \rightarrow B$ in \mathbf{X} satisfying the following four conditions:

$$(R.1) \quad f \overline{f}_A = f \text{ for all } f$$

$$(R.2) \quad \overline{f}_A \overline{g}_A = \overline{g}_A \overline{f}_A \text{ for all } \text{dom}(f) = \text{dom}(g)$$

$$(R.3) \quad \overline{g}_A \overline{f}_A = \overline{g}_A \overline{f}_A \text{ for all } \text{dom}(f) = \text{dom}(g)$$

$$(R.4) \quad \overline{g}_A f = f \overline{(gf)}_B \text{ for all } \text{cod}(f) = \text{dom}(g)$$

A category equipped with a restriction structure is called a *restriction category*.

Definition

A restriction category \mathbf{X} is called an *inverse restriction category*, whenever every map f is a restricted isomorphism. That is, each map f has a corresponding map f° such that $f^\circ f = \bar{f}$ and $ff^\circ = \overline{f^\circ}$.

properties:

- existence of pseudoinverses
- idempotents are exactly the restriction idempotents and commute

i.e., these work exactly as desired.

Question: What results in inverse semigroup theory can be extended to inverse restriction categories?

Example: Vagner-Preston works (Cockett and Lack, 2002, Thm 3.8)

Theorem (Ehresmann-Schein-Nambooripad)

The category of inductive groupoids is equivalent to the category of inverse semigroups.

Does this translate to the category of inverse restriction categories, IRCat?

Definition

A groupoid (G, \circ) is said to be an *ordered groupoid* whenever there is a partial order \leq on its arrows satisfying the following conditions:

- (i) For all arrows $f, g \in G$, $f \leq g$ implies $f^{-1} \leq g^{-1}$.
- (ii) For all arrows $a, A, b, B \in G$ such that if $a \leq A$, $b \leq B$ and the composites ab and AB exist, then $ab \leq AB$.
- (iii) For all arrows $f : A' \rightarrow B$ in G and objects $A \leq A'$ in G , there exists a unique *restriction of f to A* $[f|_*A]$ such that $\text{dom}[f|_*A] = A$ and $[f|_*A] \leq f$.

Definition

An ordered groupoid is said to be

- an *inductive groupoid* whenever its objects form a meet-semilattice,
- a *locally inductive groupoid* whenever there is a partition $\{M_i\}_{i \in I}$ of \mathbf{G}_0 into meet-semilattices M_i .

A locally inductive groupoid is said to be *top-heavy* whenever each meet-semilattice M_i admits a top-element \top_i .

Notation

Let A be an object of a restriction category \mathbf{X} . Let E_A denote the set of restrictions of all endomorphisms on A . That is,

$$E_A = \{\overline{f}_A : A \rightarrow A \mid f : A \rightarrow A \in \mathbf{X}_1\}.$$

Proposition

For each object A of a restriction category \mathbf{X} , E_A is a meet-semilattice with meets given by $\overline{a} \wedge \overline{b} = \overline{a\overline{b}}$. In addition, E_A has top element 1_A .

Remark

A restriction category is naturally partially equipped with the partial order

$$f \leq g \text{ if and only if } g\bar{f} = f$$

Construction

Given an inverse restriction category \mathbf{X} , define a groupoid $\mathcal{G}(\mathbf{X})$:

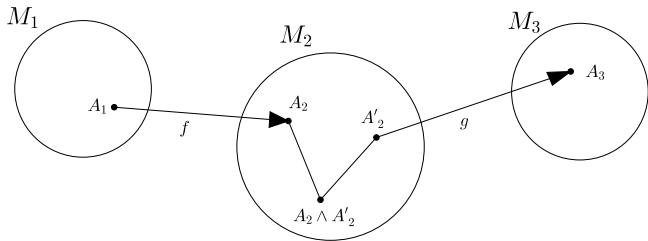
- Objects: $\mathcal{G}(\mathbf{X})_0 = \coprod_{A \in \mathbf{X}_0} E_A$.
- Arrows: $f : A \rightarrow B$ in \mathbf{X} corresponds to $f : \bar{f} \rightarrow \bar{f}^\circ$ in $\mathcal{G}(\mathbf{X})$.
 - Composition: composition in \mathbf{X}
 - Inverses: from restricted isomorphisms in \mathbf{X} .

$\mathcal{G}(\mathbf{X})$ is a top-heavy locally inductive groupoid:

- E_A meet semilattice with top 1_A
- partial order from \mathbf{X} – satisfies conditions (i) and (ii)
- restriction : $[f|_*\bar{e}] = f \circ \bar{e}$

Any two maps f and g in an ordered groupoid with $\text{dom}f \wedge \text{cod}g$ existing, there is a tensor product:

$$f \otimes g = [f \mid_* \text{dom}(f) \wedge \text{cod}(g)] \bullet [\text{dom}(f) \wedge \text{cod}(g) \mid_* g]$$



Composition in \mathbf{X} is exactly \otimes in $\mathcal{G}\mathbf{X}$.

Construction

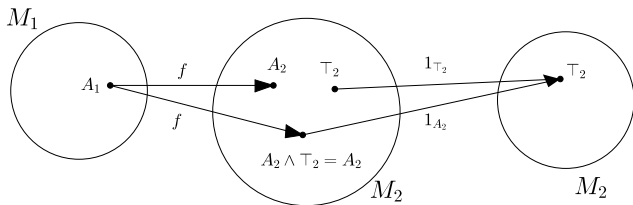
Given a top-heavy locally inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i \in I})$, define an inverse restriction category $(\mathcal{I}(\mathbf{G}), \circ, \overline{(-)})$:

- Objects: The objects are the meet-semilattices M_i .
- Arrows:

$$\mathcal{I}(\mathbf{G})(M_1, M_2) = \{f : A_1 \rightarrow A_2 \text{ in } \mathbf{G} \mid A_1 \in M_1, A_2 \in M_2\}$$

- Composition given by \otimes

Identities: Identities on the tops : $1_M = 1_{\top_M}$



$\mathcal{I}(\mathbf{G})$ is an inverse restriction category:

- Restrictions: Given an arrow $f : M_1 \rightarrow M_2$ corresponding to an arrow $f : A_1 \rightarrow A_2$ in \mathbf{G} , define

$$\bar{f} = 1_{A_1} : A_1 \rightarrow A_1$$

- Partial Inverses: For each arrow $f : M_1 \rightarrow M_2$, define

$$(f^\circ : M_2 \rightarrow M_1) = (f^{-1} : A_2 \rightarrow A_1)$$

Theorem

The functors \mathcal{G} and \mathcal{I} form an equivalence

$$\text{IRCat} \begin{array}{c} \xrightarrow{\mathcal{G}} \\ \xleftarrow{\mathcal{I}} \end{array} \text{TLIGrpd}$$

To do:

Define: double inverse semigroup is a single-object double inverse restriction category.

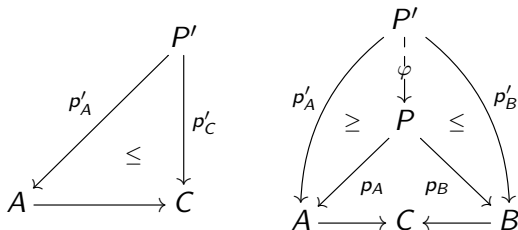
Q: What are double restriction categories and how should their restriction structures interact?

A: Unfortunately, no definition yet, but an example.

First, need restricted pullbacks in restriction category \mathbf{X} :

Given any cospan $A \longrightarrow C \longleftarrow B$, a restricted pullback is cone consisting of an object P and total arrows $p_{A,B,C} : P \rightarrow A/B/C$ satisfying the following universal property:

For each lax cone (P', p'_A, p'_B, p'_C) over $A \longrightarrow B \longleftarrow C$, there is a unique $\varphi : P' \rightarrow P$ such that $\varphi \circ p \leq p'$ and $\overline{\varphi} = \overline{p'_A} \overline{p'_B} \overline{p'_C}$



Let \mathbf{X} be a restriction category. A collection \mathcal{M} of monics in \mathbf{X} is *stable under restricted pullbacks* whenever:

- \mathcal{M} contains all isomorphisms of \mathcal{M} ,
- \mathcal{M} is closed under composition,
- for each $m : B \rightarrow C$ in \mathcal{M} and $f : A \rightarrow C$ in \mathbf{X} , the restricted pullback

$$\begin{array}{ccccc}
 A \otimes_C B & \xrightarrow{p_2} & B & & \\
 \downarrow p_1 & & \downarrow m & & \\
 A & \xrightarrow{f} & C & &
 \end{array}$$

of m along f exists and $p_1 \in \mathcal{M}$.

Define a restriction category $\text{Par}(\mathbf{X}, \mathcal{M})$ (Cockett and Lack, 2002) with the following data:

- Objects: Same objects as \mathbf{X}
- Arrows: Isomorphism classes of spans

$$X \longleftarrow^i D \longrightarrow^f Y ,$$

with $i \in \mathcal{M}$.

- Composition: restricted pullback
- Restriction: $\overline{(i, f)} = (i, i)$

Double Category $\mathbb{P}\text{ar}(\mathbf{X}, \mathcal{M})$

- Objects: Same as \mathbf{X}
- Vertical arrows: The total arrows of \mathbf{X}
 - total maps form a subcategory so composition is clear
- Horizontal arrows: the arrows of $\mathbb{P}\text{ar}(\mathbf{X}, \mathcal{M})$
 - composition restricted pullbacks
- Double cells:

$$\begin{array}{ccccc} X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\ \downarrow u \bullet & \geq & \downarrow \alpha & \leq & \downarrow \bullet v \\ X' & \xleftarrow{i'} & D' & \xrightarrow{f'} & Y' \end{array}$$

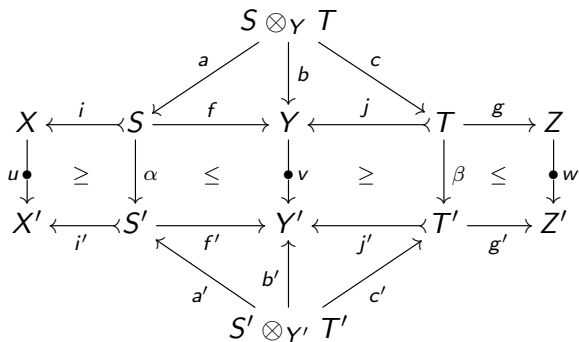
Double Cell Composition

Vertical Composition : compose all arrows vertically – straightforward

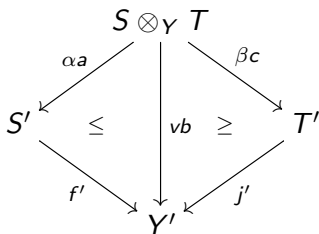
Horizontal Composition: given by universal property of restricted pullback

$$\begin{array}{ccccccc} X & \xleftarrow{i} & S & \xrightarrow{f} & Y & \xleftarrow{d} & T & \xrightarrow{x} & Z \\ \downarrow u & \geq & \downarrow \alpha & \leq & \downarrow v & \geq & \downarrow \beta & \leq & \downarrow w \\ X' & \xleftarrow{j} & S' & \xrightarrow{g} & Y' & \xleftarrow{c} & T' & \xrightarrow{y} & Z' \end{array}$$

First take the restricted pulbacks:



This gives a lax cone



over

$$S' \xrightarrow{f'} Y' \longleftarrow j' T'$$

So there is a unique $\varphi : S \otimes_Y T \rightarrow S' \otimes_{Y'} T'$ giving the double cell

$$\begin{array}{ccccc}
 X & \xleftarrow{ia} & S \otimes_Y T & \xrightarrow{gc} & Z \\
 \downarrow u \bullet & \geq & \downarrow \varphi & \leq & \downarrow \bullet w \\
 X' & \xleftarrow{i'a'} & S' \otimes_{Y'} T' & \xrightarrow{g'c'} & Z'
 \end{array}$$

For each such α , define the vertical restriction $\tilde{\alpha}$ of α to be

$$\tilde{\alpha} = \bar{u}=1_X \begin{array}{ccccc} X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\ \parallel & & \downarrow \bar{\alpha} & & \parallel \\ X & \xleftarrow{i} & D & \xrightarrow{f} & Y \end{array} \bar{v}=1_Y$$

Horizontal Restriction

For each such α , define the horizontal restriction $\bar{\alpha}$ of α to be

$$\bar{\alpha} = \begin{array}{ccccc} X & \xleftarrow{i} & D & \xrightarrow{i} & X \\ \downarrow u \bullet & \geq & \downarrow \alpha & \leq & \downarrow \bullet u \\ X' & \xleftarrow{j} & D' & \xrightarrow{j} & X' \end{array}$$

It is quickly seen that the restriction structures commute:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\
 \downarrow u & \geq & \downarrow \alpha & \leq & \downarrow v \\
 X' & \xleftarrow{i'} & D' & \xrightarrow{f'} & Y'
 \end{array} & \xrightarrow{(\widetilde{-})} & \begin{array}{ccccc}
 X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\
 \parallel & \geq & \downarrow \bar{\alpha} & \leq & \parallel \\
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 X & \xleftarrow{i} & D & \xrightarrow{i} & X \\
 \downarrow u & \geq & \downarrow \alpha & \leq & \downarrow u \\
 X' & \xleftarrow{j} & D' & \xrightarrow{j} & X'
 \end{array} & \xrightarrow{(\widetilde{-})} & \begin{array}{ccccc}
 X & \xleftarrow{i} & D & \xrightarrow{i} & X \\
 \parallel & \geq & \downarrow \bar{\alpha} & \leq & \parallel \\
 X & \xleftarrow{i} & D & \xrightarrow{i} & X
 \end{array}
 \end{array}$$

Thank you!