On Double Inverse Semigroups
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In his 2006 paper “Note on commutativity in double semigroups and two-fold monoidal categories”, Kock introduced the notion of a double semigroup, along with some commutativity properties of them. In particular, he defines double inverse semigroups.
Definition

A *double semigroup* \((S, \odot, \circlearrowleft)\) is a set equipped with two associative binary operations satisfying the *middle-four interchange law*: for all \(a, b, c, d \in S\),

\[(a \odot b) \circlearrowleft (c \circlearrowleft d) = (a \circlearrowleft c) \odot (b \circlearrowleft d).\]

- **Horizontal product**: \(a \odot b = \begin{array}{c|c} a & b \end{array} \).
- **Vertical product**: \(a \circlearrowleft b = \begin{array}{c} a \\ \hline \end{array} \). 
- **Middle-four**: 
  \[
  \begin{array}{c|c} a & b \\ \hline c & d \end{array}
  \]
Example

Any set $D$ can be made into a double semigroup by equipping it with left and right projection:

$$a \circ b = a \text{ and } a \circ b = b.$$  

Associative:

$$\begin{array}{ccc}
  a & b & c \\
  a & b & c
\end{array} = \begin{array}{c}
  c
\end{array} \quad \begin{array}{c}
  a
\end{array} = \begin{array}{c}
  a
\end{array}$$  

Middle-four interchange law:

$$\begin{array}{ccc}
  a & b \\
  c & d
\end{array} = \begin{array}{c}
  b
\end{array}$$
Definition

Given an element, $x$ in a semigroup $(S, \circ)$, $x$ said to have an inverse $x^\circ$ if

$$x = x \circ x^\circ \circ x \text{ and } x^\circ = x^\circ \circ x \circ x^\circ.$$ 

A semigroup is said to be an inverse semigroup if every element has a unique inverse. A double semigroup is said to be inverse if both of its operations are.

Note

$x \circ x^\circ$ and $x^\circ \circ x$ are idempotents:

- $(x \circ x^\circ) \circ (x \circ x^\circ) = (x \circ x^\circ \circ x) \circ x^\circ = x \circ x^\circ$
- $(x^\circ \circ x)(x^\circ \circ x) = (x^\circ \circ x \circ x \circ x^\circ) \circ x = x^\circ \circ x$
Theorem (Kock)

*Double inverse semigroups are commutative.*

In the comments of his LaTeX source code, Kock mentions that he does not have any “significant” examples of a double inverse semigroup. We aim to either find one, or to characterise double inverse semigroups.
Come soon:

- Explore Lawson’s correspondence between inductive groupoids and inverse semigroups given by a pair of constructions.
- Define double inductive groupoids.
- Extend these constructions to double inductive groupoids and double inverse semigroups and establish an analogous correspondence.
A quick notational note:

If \( f : A \rightarrow B \) is an arrow in a category:

- **Domain of** \( f \): \( f\text{dom} = A \).
- **Codomain of** \( f \): \( f\text{cod} = B \).
- **Denote the composite**

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

as \( f \circ g \) or \( fg \).
Definition

Let \((G, \bullet)\) be a groupoid and let \(\leq\) be a partial order defined on the arrows of \(G\). We call \((G, \bullet, \leq)\) and *ordered groupoid* whenever the following conditions are satisfied:

- If \(x \leq y\), then \(x^{-1} \leq y^{-1}\).
- If \(x \leq y\), \(u \leq v\), then \(xu \leq yv\).

Note

*Identification of identity arrows with objects:*

- *Gives \(\leq\) on objects*
Definition (cont’d)

- Let \( f \in G_1 \) and let \( e \) be an object in \( G \) such that \( e \leq f \text{dom} \). Then there is a unique element \((e_*|f) \in G_1\), called the restriction of \( f \) by \( e \), such that \((e_*|f) \leq f\) and \((e_*|f) \text{dom} = e\).

- Let \( f \in G_1 \) and let \( e \) be an object in \( G \) such that \( e \leq f \text{cod} \). Then there is a unique element \((f|_*e) \in G_1\), called the corestriction of \( f \) by \( e \), such that \((f|_*e) \leq f\) and \((f|_*e) \text{cod} = e\).

\[
\begin{array}{c}
\text{f dom} \xrightarrow{f} f \text{cod} \\
\downarrow \ \\
\text{e} \xrightarrow{(e_*|f)} (e_*|f) \text{cod}
\end{array}
\]
Example

Let $A$ be a set. Construct an inductive groupoid with the following data:

- Objects : $\mathcal{P}A$
- Arrows: Partial isomorphisms $f : U \simarrow V$ between subsets $U, V \in \mathcal{P}A$
- $(f : U \rightarrow V) \leq (f' : U' \rightarrow V')$ if and only if $U \subseteq U'$ and $f'$ restricted to $U$ (as functions) is $f$. 
Introduction

Inductive Groupoids and Inverse Semigroups

Quick Introduction to Double Categories

Double Inductive Groupoids and Double Inverse Semigroups

Main Result

Inductive Groupoids

Constructing Inductive Groupoids

Constructing Inverse Semigroups

An Isomorphism of Categories

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On Double Inverse Semigroups
Definition

An ordered groupoid $G$ is an *inductive groupoid* if its objects form a meet-semilattice.
Construction

*Given an inverse semigroup* $(S, \circledast)$ *with the natural partial ordering* $\leq$, *define an inductive groupoid*, $(IG(S), \bullet)$, *with the following data:*

**Objects:** *idempotents of* $S$; $IG(S)_0 = E(S)$.

**Arrows:** *elements of* $S$. 
Construction (cont’d)

**Arrows:** *elements of S.*

- $s\text{dom} = s \circ s^\circ$
- $s\text{cod} = s^\circ \circ s$
- If $a^\circ \circ a = b \circ b^\circ$, define $a \bullet b = a \circ b$
- Every arrow is an isomorphism with $a^{-1} = a^\circ$
- $(a|_e) = a \circ e$
- $(e_*|a) = e \circ a$
Given an inductive groupoid \((G, \bullet, \leq, \wedge)\), construct an inverse semigroup \((\text{IS}(G), \odot)\) with \(\text{IS}(G) = G_1\) and, for any \(a, b \in S\),

\[a \odot b = (a|_* \text{acod} \wedge \text{bdom}) \bullet (\text{acod} \wedge \text{bdom}_* \vert b).\]
**Notation**

Denote the category of inverse semigroups and semigroup homomorphisms as \( \text{IS} \). Denote the category of inductive groupoids and inductive functors as \( \text{IG} \).

**Theorem (Lawson)**

The categories \( \text{IG} \) and \( \text{IS} \) are isomorphic.

**GOAL:** Double this theorem
A *double category* $\mathcal{D}$ consists of the following data:

- A collection $\mathcal{D}_0$ of objects.
- A collection $\text{Ver}(\mathcal{D})$ of vertical arrows.

Associative and unitary composition:

$$A \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{f \cdot g} C$$

$$A \xrightarrow{1_A} A \xrightarrow{f} B = A \xrightarrow{f} B = A \xrightarrow{f} B \xrightarrow{1_B} B$$
Definition (cont’d)

– A collection $\text{Hor}(\mathcal{D})$ of horizontal arrows. 
  Associative and unitary composition:

\[
\begin{align*}
A & \xrightarrow{f} B \xrightarrow{g} C = A \xrightarrow{f \circ g} C \\
A & \xrightarrow{\text{id}_A} A \xrightarrow{f} B = A \xrightarrow{f} B = A \xrightarrow{f} B \xrightarrow{\text{id}_B} B
\end{align*}
\]
Definition (Cont’d)

– A collection $\text{Db1}(\mathcal{D})$ of double cells. A double cell $\alpha$ has the following form:

$$
\begin{array}{c}
A \xrightarrow{f} B \\
\downarrow \alpha \downarrow \\
C \xrightarrow{g} D
\end{array}
$$

- $A, B, C$ and $D$ are objects of $\mathcal{D}$.
- Horizontal domain and codomain:

$$
\alpha_{\text{hdom}} = u \text{ and } \alpha_{\text{hcod}} = v
$$

- Vertical domain and codomain:

$$
\alpha_{\text{vdom}} = f \text{ and } \alpha_{\text{vcod}} = g
$$
Definition (cont’d)

These double cells must come together with:

- An associative and unitary horizontal composition, $\circ$.
- An associative and unitary vertical composition, $\bullet$.
- Horizontal and vertical composition of double cells must satisfy the middle-four interchange law. That is, for any $\alpha, \beta, \gamma, \delta \in \text{Dbl}(\mathcal{D})$,

$$ (\alpha \bullet \beta) \circ (\gamma \bullet \delta) = (\alpha \circ \gamma) \bullet (\beta \circ \delta). $$
A double inductive groupoid, denoted DIG,

\[ \mathcal{G} = (\text{Obj}(\mathcal{G}), \text{Ver}(\mathcal{G}), \text{Hor}(\mathcal{G}), \text{Dbl}(\mathcal{G})) \]

is a double groupoid (i.e., a double category in which every vertical arrow, horizontal arrow and double cell is an isomorphism) such that:
Definition (cont’d)

\((\text{Ver}(\mathcal{G}), \text{Dbl}(\mathcal{G}))\) is an inductive groupoid.

- Composition: horizontal composition, \(\circ\).
- Partial order: \(\leq\).
- Meet of vertical arrows \(e\) and \(f\): \(e \wedge_h f\).
- For a double cell \(\alpha\) and a vertical arrow \(e\) with \(e \leq \alpha h\text{dom}\), horizontal restriction: \((e \ast |\alpha)\).
- If \(e \leq \alpha h\text{cod}\), horizontal corestriction: \((\alpha | \ast e)\).
Definition (cont’d)

\((\text{Hor}(\mathcal{G}), \text{Dbl}(\mathcal{G}))\) is an inductive groupoid.

- Composition: vertical composition, \(\bullet\).
- Partial order: \(\preceq\).
- Meet of horizontal arrows \(e\) and \(f\): \(e \land_v f\).
- For a double cell \(\alpha\) and a horizontal arrow \(e\) with \(e \preceq \alpha \text{vdom}\), vertical restriction: \([e_*|\alpha]\).
- If \(e \preceq \alpha \text{vcod}\), vertical corestriction: \([\alpha_*|e]\).
Definition (cont’d)

If $a, b$ are double cells, $f’, g’$ are horizontal arrows and $f, g$ are vertical arrows, the following laws about restrictions and corestrictions preserving composition hold:

1. $(a \bullet b |_f \bullet g) = (a |_f) \bullet (b |_g)$.
2. $[a \circ b |_f' \circ g'] = [a |_f'] \circ [b |_g']$.
3. $(f \bullet g |_a \bullet b) = (f |_a) \bullet (g |_b)$.
4. $[f' \circ g' |_a \circ b] = [f' |_a] \circ [g' |_b]$. 
\[(a \cdot b|_\ast f \cdot g) = (a|_\ast f) \bullet (b|_\ast g)\]
Definition (cont’d)

If $e, f, g$ and $h$ are horizontal arrows and $e', f', g'$ and $h'$ are vertical arrows, the following laws about composition and meets satisfying middle-four hold:

(a) $(e \land_v f) \circ (g \land_v h) = (e \circ g) \land_v (f \circ h)$.

(b) $(e' \land_h f') \bullet (g' \land_h h') = (e' \bullet g') \land_h (f' \bullet h')$. 
\[(e \land_h f) \cdot (g \land_h h) = (e \cdot g) \land_h (f \cdot h)\]
Definition (cont’d)

If \( e \) and \( g \) are horizontal arrows \( f \) and \( h \) are objects, then the following rule about corestrictions and meets satisfying middle-four holds:

\[
(e|_*f) \land_V (g|_*h) = (e \land_V g|_*f \land_V h)
\]
(e|_f) \land_V (g|_h) = (e \land_V g|_f \land_V h)

\[
\begin{array}{ccc}
(e|_f) & \rightarrow & f \\
\land_V & = & \rightarrow \\
(g|_h) & \rightarrow & h
\end{array}
\]

Similarly,

(a) \((e|_f) \land_V (g|_h) = (e \land_V g|_f \land_V h)\).

(b) \([e'|_f'] \land_h [g'|_h'] = [e' \land_h g'|_f' \land_h h']\).

(c) \((e|_f) \land_V (g|_h) = (e \land_V g|_f \land_V h)\).

(d) \([e'|_f'] \land_h [g'|_h'] = [e' \land_h g'|_f' \land_h h']\).
Definition (cont’d)

If $a$ is a double cell, $f$ a horizontal arrow, $g$ a vertical arrow and $x$ an object such that

\[
\begin{align*}
  f & \preceq avcod \\
  g & \preceq ahcod \\
  x & = f hcod \land g vcod,
\end{align*}
\]

then the following middle-four law about vertical and horizontal corestrictions holds:

\[
([a f^*] [g x^*]) = ([a g^*] [f x^*])
\]
\[(a|f)\ast (g\ast x) = (a|g)\ast (f\ast x)\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
a
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
f
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Similarly,

(a) \([(a|g)\ast (f\ast x)] = (a|f)\ast (g\ast x)\).

(b) \([(x|g)\ast (f\ast a) = (x|f)\ast (g\ast a)\).

(c) \([(x|f)\ast (g\ast a) = (x|g)\ast (f\ast a)].\)
Definition (cont’d)

If \( e, f, g \) and \( h \) are objects, the following law about meets satisfying middle-four holds:

\[
(e \land_h f) \land_v (g \land_h h) = (e \land_v g) \land_h (f \land_v h).
\]
Definition (cont’d)

If $a$ is a double cell, $e$ a vertical arrow and $e'$ a horizontal arrow, then the following laws about domains and codomains preserving restrictions and corestrictions hold:

(a) $(a\ |\ e)v\downarrow = (av\downarrow | e\downarrow v)$. 
(b) $(a\ |\ e)v\downarrow = (av\downarrow | e\downarrow v)$. 
(c) $(e\ |\ a)v\downarrow = (ev\downarrow | a\downarrow v)$. 
(d) $(e\ |\ a)v\downarrow = (ev\downarrow | a\downarrow v)$. 
(e) $[a\ |\ e']h\downarrow = [ah\downarrow | e'h\downarrow]$. 
(f) $[a\ |\ e']h\downarrow = [ah\downarrow | e'h\downarrow]$. 
(g) $[e'\ |\ a]h\downarrow = [e'\ |\ a]h\downarrow$. 
(h) $[e'\ |\ a]h\downarrow = [e'\ |\ a]h\downarrow$. 

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\[
(a|e)_{v\text{dom}} = (a_{v\text{dom}}|e_{v\text{dom}}) \\
(a|e)_{h\text{dom}} = (a_{h\text{dom}}|e_{h\text{dom}})
\]
Construction (DIG)

Given a double inverse semigroup \((S, \circlearrowleft, \circlearrowright)\), we construct a double inductive groupoid

\[
\text{DIG}(S) = (\text{DIG}(S)_0, \text{Ver}(\text{DIG}(S)), \text{Hor}(\text{DIG}(S)), \text{Dbl}(\text{DIG}(S)))
\]

as follows:

Objects: \(\text{DIG}(S)_0 = E(S, \circlearrowleft) \cap E(S, \circlearrowright)\).
Construction (DIG cont’d)

**Vertical arrows:** $\text{Ver}(\text{DIG}(S)) = E(S, \odot)$. Let $u$ and $v$ be any two vertically composable arrows:

- $uv\text{dom} = u \odot u^\circ$
- $uv\text{cod} = u^\circ \odot u$
- Vertical composition: $u \bullet v = u \odot v$

**Horizontal arrows:** $\text{Hor}(\text{DIG}(S)) = E(S, \odot)$. Let $f$ and $g$ be any two horizontally composable arrows:

- $fh\text{dom} = f \odot f^\circ$
- $fh\text{cod} = f^\circ \odot f$
- Horizontal composition: $f \circ g = f \odot g$
Construction (DIG cont’d)

**Dbl(DIG(S)) = S(\odot, \odot).** Let \(a, b\) be any two **horizontally composable double cells**.

**Horizontally:**

- \(ah\text{dom} = a \odot a\)
- \(ah\text{cod} = a \odot a\)
- **Horizontal composition:** \(a \circ b = a \odot b\)
- **Horizontal partial order:** \(a \leq b\) iff \(a = \text{id}_e \odot b\) for some **vertical arrow** \(e\)
- **Horizontal meet of two vertical arrows** \(e\) and \(f\): \(e \land_h f = e \odot f\)
- **If we have a vertical arrow** \(e \leq ah\text{cod}\), define \((a | e) = a \odot e\)
- **If** \(e \leq ah\text{dom}\), define \((e | a) = e \odot a\). 

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On Double Inverse Semigroups
Construction (DIG cont’d)

\[ \text{Dbl}(\text{DIG}(S)) = S(\odot, \odot). \] Let \( a, b \) be any two vertically composable double cells.

**Vertically:**
- \( \text{avdom} = a \odot a^\circ \)
- \( \text{avcod} = a^\circ \odot a \)
- **Vertical composition:** \( a \bullet b = a \odot b \)
- **Vertical partial order:** \( a \preceq b \) iff \( a = 1^e \odot b \) for some horizontal arrow \( e \)
- **Vertical meet of two horizontal arrows** \( e \) and \( f \): \( e \land_v f = e \odot f \)
- **If we have a horizontal arrow** \( e \preceq \text{avcod} \), define \( [a|_e^*] = a \odot e \)
- **If** \( e \preceq \text{avdom} \), define \( [e^*_a] = e \odot a \)
Double Inductive Groupoids from Double Inverse Semigroups

**Theorem**

*If $S(⊙, □)$ is a double inverse semigroup, then $\text{DIG}(S)$, as constructed in Construction DIG, is a double inductive groupoid.*
Double Inverse Semigroups from Double Inductive Groupoids

Construction (DIS)

Given a double inductive groupoid

\[ \mathcal{G} = (\text{Obj}(\mathcal{G}), \text{Ver}(\mathcal{G}), \text{Hor}(\mathcal{G}), \text{Dbl}(\mathcal{G})) \],

we construct a double inverse semigroup \( \text{DIS}(\mathcal{G}) = (S, \odot, \circlearrowright) \) as follows:

- Its elements are the double cells of \( \mathcal{G} \); \( S = \text{Dbl}(\mathcal{G}) \).
Construction (DIS cont’d)

- For any \( a, b \in S \), define

\[
a \odot b = (a|_* ah\text{cod} \wedge_h bh\text{dom}) \circ (ah\text{cod} \wedge_h bh\text{dom}_*|b)
\]

- For any \( a, b \in S \), define

\[
a \odot b = [a|_* av\text{cod} \wedge_v bv\text{dom}] \bullet [av\text{cod} \wedge_v bv\text{dom}_*|b]
\]
Double Inverse Semigroups from Double Inductive Groupoids

**Theorem**

If $G$ is a double inductive groupoid, then $\text{DIS}(G)$, as constructed in Construction DIS, is a double inverse semigroup.

Most of the work in proving this is in checking that the middle-four interchange law is satisfied.
An Isomorphism of Categories

Notation

We denote the category of double inductive groupoids with double inductive functors as $\text{DIG}$ and we denote the category of double inverse semigroups with double semigroup homomorphisms as $\text{DIS}$.

Theorem

There exists an isomorphism of categories between $\text{DIG}$ and $\text{DIS}$. 
Consider a double cell in a double inductive groupoid

\[
\begin{array}{c}
A \xymatrix{ & B \ar[ld]_{a} \ar[rd]^a \ar[d]^{a \text{hdom}} \ar[ru]_{a \text{vdom}} \ar[ru]^{a \text{hcod}} \ar[d]^{a \text{vcod}} \\
C & & D
\end{array}
\]
Recall that domains and codomains may be written as semigroup products and that double inverse semigroups are commutative. Then

\[ a_h := \text{ahdom} = a \odot a^\circ = a^\circ \odot a = \text{ahcod} \]

\[ a_v := \text{avdom} = a \odot a^\circ = a^\circ \odot a = \text{ahcod} \]
Similarly, the domain and codomain of a vertical or horizontal arrows are equal, so that

- $A = a_h \text{hdom} = a_h \text{hcod}$
- $A = a_v \text{vdom} = a_v \text{vcod}$

Ultimately, $a$ is of the form

$$
\begin{align*}
A & \xrightarrow{a_v \text{dom}} B \\
\downarrow a_{\text{hdom}} & \quad \quad \quad \quad \downarrow a_{\text{hcod}} \\
C & \xrightarrow{a_v \text{cod}} D \\
\downarrow a_{\text{vdom}} & \quad \quad \quad \quad \downarrow a_{\text{vcod}} \\
A & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
\end{align*}
$$
Let $\mathcal{G}$ be a double inductive groupoid and let $A$ be an object of $\mathcal{G}$. Then there is a natural collection of double cells

$$(A)S_\mathcal{G} = \left\{ a \in \text{Dbl}(\mathcal{G}) \mid \begin{array}{c} A \xrightarrow{a_h} A \\ a \downarrow \quad a \downarrow \\ A \xrightarrow{a_h} A \end{array} \right\}$$
Recall: Objects of double inductive groupoids are idempotent with respect to both operations of its corresponding double inverse semigroup.

Double inverse semigroups are commutative.

\[(a \odot b) \odot (a \odot b) = (a \odot a) \odot (b \odot b) = a \odot b\]

\[a \odot b = (a \odot b) \odot (a \odot b)\]
\[= (a \odot b) \odot (b \odot a)\]
\[= (a \odot b) \odot (b \odot a)\]
\[= (a \odot b) \odot (a \odot b)\]
\[= a \odot b.\]
Lemma

The vertical and horizontal order relations on the objects of a double inductive groupoid coincide.
Theorem

These one-object double inductive groupoids are precisely Abelian groups.

Proposition

Let $G$ be a double inductive groupoid. If $A$ and $B$ are objects in $G$ with $A \leq B$, then there is an Abelian group homomorphism

$$\varphi_{A \leq B} : (B)_G \rightarrow (A)_G.$$
This discussion results in an $\mathbf{Ab}$–valued presheaf

\[ S_g : \text{Obj}(G)^{\text{op}} \to \mathbf{Ab}. \]

**Theorem**

*Arbitrary double inverse semigroups are $\mathbf{Ab}$–valued presheaves over meet-semilattices.*
Construction

If \( P : \text{L}^{\text{op}} \to \text{Ab} \) is a presheaf of Abelian groups on a meet-semilattice, define a double inductive groupoid \( \mathcal{G} = PF' \) with the following data:

Objects: \( \text{Obj}(\mathcal{G}) = \text{L} \)

Vertical/horizontal arrows:
\( \text{Ver}(\mathcal{G}) = \text{Hor}(\mathcal{G}) = \{ e_A : A \to A : A \in \text{L} \} \),
- \( e_A \) is the group unit of the Abelian group \( AP \) for each \( A \) in \( \text{L} \).
- (Co)domains: \( e_A \text{dom} = e_A \text{cod} = A \)
- Composition: \( e_A \circ e_A = e_A \bullet e_A = e_A \).
- Meets: \( e_A \land e_B = A \land B \) to be that from \( \text{L} \).
Construction (cont’d)

Double cells: \( \text{Dbl}(G) = \coprod_{A \in L} AP \)

- Disjoint union of all Abelian groups \( AP \) for \( A \) in \( L \).
- A double cell \( a \) is contained in an Abelian group \( AP \) for some \( A \in L \).
- \( ah\text{dom} = ah\text{cod} = av\text{dom} = v\text{cod} = e_A \).
- Composites: group products
- If \( e_u \leq e_A = ah\text{dom} \),
  - Restriction of \( a \) to \( e_u \):
    \[
    (e_u^* | a) = e_u *_u (a) \varphi_{u \leq A} = (a) \varphi_{u \leq A}
    \]
  - Corestrictions are similarly defined.
Notation

Denote the category of presheaves of Abelian groups on meet-semilattices by \( \text{AbMeetSLatt} \).

Theorem

The categories \( \text{DIG} \) and \( \text{AbMeetSLatt} \) are isomorphic.
Recall:

- Kock showed double inverse semigroups are commutative.
- Double inverse semigroups are exactly presheaves of Abelian groups on meet-semilattices.

**Theorem**

*Double inverse semigroups are commutative and improper. That is, \((S, \circ, \Diamond)\) is a double inverse semigroup if and only if both \(\circ\) and \(\Diamond\) are commutative inverse semigroup operations with \(\Diamond = \Diamond\).*