

An Element-based Reformulation of Restriction Monads

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Restriction Categories

A restriction structure on a category \mathbf{X} is an assignment of an arrow $\bar{f} : A \rightarrow A$ to each arrow $f : A \rightarrow B$ in \mathbf{X} satisfying the following four conditions:

(R.1) For all maps f , $f\bar{f} = f$.

(R.2) For all maps $f : A \rightarrow B$ and $g : A \rightarrow B'$, $\bar{f}\bar{g} = \bar{g}\bar{f}$.

(R.3) For all maps $f : A \rightarrow B$ and $g : A \rightarrow B'$, $\overline{g\bar{f}} = \bar{g}\bar{f}$.

(R.4) For all maps $f : B \rightarrow A$ and $g : A \rightarrow B'$, $\bar{g}f = f\overline{g\bar{f}}$.

A category equipped with a restriction structure is called a *restriction category*.

Restriction Monads: Definition Version 1

In a bicategory with involution, a restriction monad consists of a 0-cell x , 1-cells $T, D, E : x \rightarrow x$ and 2-cells

- $\eta : 1_T \Rightarrow T$,
- $\mu : T^2 \Rightarrow T, [\mu |_* DE] : DE \Rightarrow D$,
- $\rho : D \Rightarrow E$ (epic),
- $\iota : E \Rightarrow T$ (monic),
- $\Delta : T \Rightarrow TD, \tau : D^2 \Rightarrow D^2$ and
- $\psi : DT \Rightarrow TD$

satisfying conditions corresponding to (R.1) through (R.4) plus the usual monad laws plus $D^*D = DD^*$

Problem with the first approach

Ordinary monads in $\text{Span}(\mathbf{Set})$ are in one-to-one correspondence with small categories.

Let \mathbf{X} be a restriction category. We can easily construct a restriction monad $R(\mathbf{X})$ in $\text{Span}(\mathbf{Set})$ with T, D, E behaving as desired, but we can't canonically go backwards: the D is not uniquely determined by the choice of T .

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One Solution:

Define restriction monads so that D and E naturally become "subobjects" of T by design. We can do this easily if we think of T as having elements and defining our operations on certain subsets of T .

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

Suppose that \mathbf{X} is a small restriction category. For each element A of \mathbf{X}_0 , we can define a span $\vec{A} : \{*\} \longrightarrow \mathbf{X}_0$ by

$$\begin{array}{ccc} & \{*\} & \\ \text{id} \swarrow & & \searrow A \\ \{*\} & & \mathbf{X}_0 \end{array}$$

Peek-ahead: We will call such a span $\{*\}$ -*elemental*.

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

Suppose that \mathbf{X} is a small restriction category. Its corresponding monad in $\text{Span}(\mathbf{Set})$ is of the form

$$\begin{array}{ccc} & \mathbf{X}_1 & \\ s \swarrow & & \searrow t \\ \mathbf{X}_0 & & \mathbf{X}_0 \end{array}$$

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

Suppose that \mathbf{X} is a small restriction category. Its corresponding monad in $\text{Span}(\mathbf{Set})$ is of the form

$$\mathbf{X}_0 \xleftarrow{s} \mathbf{X}_1 \xrightarrow{t} \mathbf{X}_0$$

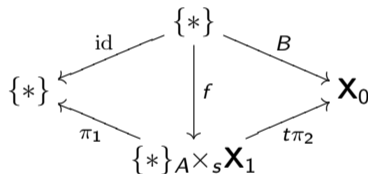
Composing \vec{A} with T , then is of the form

$$\{\ast\} \xleftarrow{\text{id}} \{\ast\} A \times_s \mathbf{X}_1 \xrightarrow{t\pi_1} \mathbf{X}_0$$

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

$T\vec{A}$ contains as data all arrows of \mathbf{X} with source A .

Given another object $B \in \mathbf{X}_0$, a span morphism $f : \vec{B} \longrightarrow T\vec{A}$, of the form



is therefore equivalent to the choice of an arrow f in \mathbf{X} whose source is A and whose target is B .

$$\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \longleftrightarrow \mathbf{X}(A, B).$$

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

Such an identification allows us to define the restriction operator ρ as a family of set functions

$$\rho_{A,B} : \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \rightarrow \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

of arrows $f : A \rightarrow B$ to arrows $\rho(f) : A \rightarrow A$.

The conditions that this family of assignments satisfies will be given in a definition soon.

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

Identifying $\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we must therefore consider how to “compose” elements of the set

$$\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \times \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}).$$

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Note that such an element is of the form

$$\begin{array}{ccc} \vec{C} & \longrightarrow & T\vec{B} \\ & & \downarrow \\ & & \vec{B} \longrightarrow T\vec{A}, \end{array}$$

For all $A, B, C \in \mathbf{X}_0$, define a composition map $\tilde{\mu}$ to be a Kleisli-flavoured composite.

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

$$\begin{array}{c}
 \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \times \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}) \\
 \downarrow \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T, T) \times \text{id} \\
 \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A}) \times \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}) \\
 \downarrow \circ_{TT\vec{A}, T\vec{B}, \vec{C}}^{\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)} \\
 \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, TT\vec{A}) \\
 \downarrow \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, \mu) \\
 \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{A})
 \end{array}$$

$\tilde{\mu}$ is the composite:

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

Defining $\tilde{\mu}$ first requires an interpretation of the set

$$\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A}).$$

Its elements are span morphisms of the form

$$\begin{array}{ccccc}
 & & \{*\} B \times_s \mathbf{X}_1 & & \\
 & \swarrow \pi_1 & \downarrow f & \searrow t\pi_2 & \\
 \{*\} & & & & \mathbf{X}_0 \\
 & \swarrow \pi_1 & & \searrow t\pi_2 \pi_2 & \\
 & & \{*\} A \times_{s\pi_1} (\mathbf{X}_1 t \times_s \mathbf{X}_1) & &
 \end{array}$$

These are assignments of arrows f with source B to composable pairs of arrows with source A and target tf :

$$(B \rightarrow C) \mapsto (A \rightarrow C' \rightarrow C).$$

Example: The elements of a monad in $\text{Span}(\mathbf{Set})$.

The morphism $\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T, T)$

$$\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \longrightarrow \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A})$$

is defined by

$$\left[f : A \rightarrow B \right] \longmapsto \left[(f, -) : (g : B \rightarrow C) \longmapsto (f : A \rightarrow B, g : B \rightarrow C) \right]$$

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We can then compute this composite $\tilde{\mu}$:

$$(f : A \rightarrow B, g : B \rightarrow C) \mapsto ((f, -), g) \mapsto (f, g) \mapsto \mu(f, g) = g \circ f;$$

The composition defined by μ coincides with $\tilde{\mu}$ in $\text{Span}(\mathbf{Set})$.

Restriction Monads: New Definition

Suppose that \mathcal{B} is a bicategory with an involution on 1-cells containing an object E satisfying $\mathcal{B}(E, E)_0 \cong \mathcal{B}_0$.

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A restriction monad in \mathcal{B} is a monad (T, η, μ) in \mathcal{B} together with a family of functions

$$\rho_{A,B} : \mathcal{B}(E, x)(B, TA) \rightarrow \mathcal{B}(E, x)(A, TA)$$

indexed by E -elemental one-cells $A, B : E \rightarrow x$.

(Definition: E -elemental means: $A^*A \cong \text{id}_E$.)

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Together with a way to compose morphisms between E -elemental one-cells, we impose four axioms on restriction monads, coming up.

Restriction Monads: New Definition

The way to compose morphisms between E -elemental one-cells is a multiplication map

$$\tilde{\mu}_{A,B,C} : \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) \rightarrow \mathcal{B}(E, x)(C, TA)$$

of E -elemental 1-cells defined by the composite

$$\left(\mathcal{B}(E, x)(T, T) \times \text{id} \right); \left(\circ_{TTA, TB, C}^{\mathcal{B}(E, x)} \right); \mathcal{B}(E, x)(C, \mu_A).$$

For every triple of 1-cells $A, B, C : 1 \rightarrow x$, we require that the following diagrams commute which correspond to (R1)–(R4):

Restriction Monads: New Definition

 (R.1) " $f\bar{f} = f$ "

$$\begin{array}{ccc}
 \mathcal{B}(E, x)(B, TA) & \xrightarrow{\Delta} & \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(B, TA) \\
 \parallel & & \downarrow \rho \times \text{id} \\
 \mathcal{B}(E, x)(B, TA) & \xleftarrow{\tilde{\mu}} & \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, x)(B, TA)
 \end{array}$$

Restriction Monads: New Definition

 (R.2) " $\bar{f} \bar{g} = \bar{g} \bar{f}$ "

$$\begin{array}{ccc}
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TA) & \xrightarrow{\rho \times \rho} & \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, x)(A, TA) \\
 \tau \downarrow & & \downarrow \tilde{\mu} \\
 \mathcal{B}(E, x)(C, TA) \times \mathcal{B}(E, x)(B, TA) & & \mathcal{B}(E, x)(A, TA) \\
 \rho \times \rho \downarrow & \nearrow \tilde{\mu} & \\
 \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, x)(A, TA) & &
 \end{array}$$

Restriction Monads: New Definition

(R.3) " $\overline{g\bar{f}} = \bar{g}\bar{f}$ "

$$\begin{array}{ccc}
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TA) & \xrightarrow{\rho \times \text{id}} & \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, x)(C, TA) \\
 \downarrow \rho \times \rho & & \downarrow \tilde{\mu} \\
 & & \mathcal{B}(E, x)(C, TA) \\
 & & \downarrow \rho \\
 \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, x)(A, TA) & \xrightarrow{\tilde{\mu}} & \mathcal{B}(E, x)(A, TA)
 \end{array}$$

Restriction Monads: New Definition

(R.4) “ $\overline{g} f = f \overline{g f}$ ”

$$\begin{array}{ccc}
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) & \xrightarrow{\text{id} \times \rho} & \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(B, TB) \\
 \downarrow \Delta \times \text{id} & & \downarrow \tilde{\mu} \\
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) & & \\
 \downarrow \text{id} \times \tilde{\mu} & & \\
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TA) & \xrightarrow{\text{id} \times \rho} & \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(A, TA) \xrightarrow{\tilde{\mu} \cdot \tau} \mathcal{B}(E, x)(B, TA)
 \end{array}$$

Restriction Monads as Internal Categories

Proposition

Small restriction categories are in one-to-one correspondence with restriction monads in $\text{Span}(\mathbf{Set})$.

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Proposition

Let \mathbf{C} be a category with all pullbacks over s and t . Restriction monads in $\text{Span}(\mathbf{C})$ are in one-to-one correspondence with restriction categories internal to \mathbf{C} .

Restriction Monads as Enriched Categories

Proposition

Restriction monads in $\mathbf{Set}\text{-Mat}$ are in one-to-one correspondence with small restriction categories.

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Proposition

If \mathcal{V} is a Cartesian monoidal category, then restriction \mathcal{V} -categories are restriction monads in $\mathcal{V}\text{-Mat}$.

Algebras for Restriction Monads

Let T be an ordinary monad in $\text{Span}(\mathbf{Set})$ and that \mathbf{X} is its corresponding small category.

Recall that algebras (S, h) for T are right- \mathbf{X} modules on the apex set of $S = \mathbf{X}_0 \xleftarrow{a} M \xrightarrow{b} Y$ with the action given by $h : ST \Rightarrow S$:

$$\begin{array}{ccccc}
 & & \mathbf{X}_1 & t \times_a & M \\
 & \swarrow & & \downarrow & \searrow \\
 & & s\pi_1 & h & b\pi_2 \\
 \mathbf{X}_0 & & & & Y \\
 & \swarrow & & \downarrow & \searrow \\
 & & a & M & b
 \end{array}$$

Similarly to how we identify the hom-set $\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we identify $\text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with the module set $S(B, A)$.

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We can then define a restriction operator r as a family of set functions

$$r_{A,B} : \text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}) \rightarrow \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

$$\alpha : A \longrightarrow B \longmapsto r(\alpha) : A \rightarrow A$$

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We can then define a restriction operator r as a family of set functions

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$$\alpha : A \longrightarrow B \longmapsto r(\alpha) : A \rightarrow A$$

We will require that each $r(\alpha)$ is a restriction idempotent of \mathbf{X} :

$$\text{Im}(r_{A,B}) \subseteq \cup_{A': 1 \rightarrow X} \text{Im}(\rho_{A,A'})$$

Identifying $\text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with $S(B, A)$, we must therefore consider how to “ h -act” with elements of the set

$$\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A}') \times \text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

Much like $\tilde{\mu}$, we will define \tilde{h} as a Kleisli-styled composite which will coincide (on the hom-categories) with h in $\text{Span}(\mathbf{Set})$.

There will also be restriction axiom diagrams which will have to commute.

$$\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A}') \times \text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

$$\downarrow \text{Span}(\mathbf{Set})(\{*\}, S\mathbf{X}_0)(S, S) \times \text{id}$$

$$\text{Span}(\mathbf{Set})(\{*\}, Y)(S\vec{A}, ST\vec{A}') \times \text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

$$\downarrow \circ_{\text{Span}(\mathbf{Set})(\{*\}, Y)} \text{Span}(\mathbf{Set})(\{*\}, Y)$$

$$ST\vec{A}', S\vec{A}, \vec{B}$$

$$\text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, ST\vec{A}')$$

$$\downarrow \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, h)$$

$$\text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}')$$

\tilde{h} is the composite:

(R.1)

$$\begin{array}{ccc}
 \mathcal{B}(E, y)(B, SA) & \xrightarrow{\Delta} & \mathcal{B}(E, y)(B, SA) \times \mathcal{B}(E, y)(B, SA) \\
 \parallel & & \downarrow r \times \text{id} \\
 \mathcal{B}(E, y)(B, SA) & \xleftarrow{\tilde{h}} & \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, y)(B, SA)
 \end{array}$$

(R.3)

$$\begin{array}{ccc}
 \mathcal{B}(E, y)(B, SA) \times \mathcal{B}(E, y)(B', SA) & \xrightarrow{r \times \text{id}} & \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, y)(B', SA) \\
 \downarrow r \times r & & \downarrow \tilde{h} \\
 & & \mathcal{B}(E, y)(B', SA) \\
 & & \downarrow r \\
 \mathcal{B}(E, x)(A, TA) \times \mathcal{B}(E, x)(A, TA) & \xrightarrow{\tilde{\mu}} & \mathcal{B}(E, x)(A, TA)
 \end{array}$$

(R.4)

$$\begin{array}{ccc}
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) & \xrightarrow{\text{id} \times r} & \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(B, TB) \\
 \Delta \times \text{id} \downarrow & & \downarrow \tilde{\mu} \\
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TB) & & \\
 \text{id} \times \tilde{h} \downarrow & & \\
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(C, TA) & & \\
 \text{id} \times r \downarrow & & \\
 \mathcal{B}(E, x)(B, TA) \times \mathcal{B}(E, x)(A, TA) & \xrightarrow{\tilde{\mu} \cdot \tau} & \mathcal{B}(E, x)(B, TA)
 \end{array}$$

Proposition

Let \mathbf{X} be a small restriction category and let (T, η, μ, ρ) denote its corresponding restriction monad in $\text{Span}(\mathbf{Set})$. An algebra

$$(S = \mathbf{X}_0 \xleftarrow{a} M \xrightarrow{b} Y, h, r)$$

is a right- \mathbf{X} restriction module, whose right \mathbf{X} -action is defined by h -evaluation.