Restriction Monads

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Restriction Categories (Cockett and Lack, 2002)

A category $\mathbf{X}$ is called a restriction category when it can be equipped with an assignment

$$(f : A \to B) \mapsto (\overline{f_A} : A \to A)$$

of all arrows f in $\mathbf{X}$ to an endomorphism $\overline{f}$ satisfying:

1. For all maps f, $f\,\overline{f_A} = f$.
2. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f_A}\,\overline{g_A} = \overline{g_A}\,\overline{f_A}$.
3. For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g_A\,\overline{f_A}} = \overline{g_A}\,\overline{f_A}$.
4. For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g_A}\,f = f\,\overline{(gf)_B}$.

Restriction Category Objects

- Obvious data needed:

$$\mathbb{C} = \mathbf{X}_1 \, {}_t\!\times_s \mathbf{X}_1 \xrightarrow{c} \mathbf{X}_1 \underset{t}{\overset{s}{\rightrightarrows}} \mathbf{X}_0, \quad u : \mathbf{X}_0 \to \mathbf{X}_1, \quad r : \mathbf{X}_1 \to \mathbf{X}_1$$

- What additional data is needed to allow us to diagrammatically express (R.1) - (R.4)?
- Also, want to keep an eye out and avoid using the fact that restriction categories are internal to **Set**.

Restriction Monads

In a bicategory, a restriction monad consists of a 0-cell x, 1-cells

$$T, D, E : x \to x$$

and 2-cells

- $\eta : 1_T \Rightarrow T$,
- $\mu : T^2 \Rightarrow T$,
- $[\mu |_* DE] : DE \Rightarrow D$,
- $\rho : D \Rightarrow E$ (epic),
- $\iota : E \Rightarrow T$ (monic),
- $\Delta : T \Rightarrow TD$,
- $\tau : D^2 \Rightarrow D^2$ and
- $\psi : DT \Rightarrow TD$

satisfying conditions corresponding to (R.1) through (R.4) plus the usual monad laws.

Example: $\mathrm{Par} : \mathbf{Set} \to \mathbf{Set}$ in **Cat**

Define a functor $\mathrm{Par} : \mathbf{Set} \to \mathbf{Set}$ by $\mathrm{Par}(A) = A \coprod \{\star\}$ and

$$\mathrm{Par}(f : A \to B)(x) = \begin{cases} f(x) & x \in A \\ \star & x = \star \end{cases}$$

A monad with

- $\eta_A : A \to A \coprod \{\star\} : a \mapsto a$
- $\mu_A : (A \coprod \{\star\}) \coprod \{\star\} \to A \coprod \{\star\}$

Its Kleisli arrows are total representations of partial functions; a partial function $f : A \to B$ can be thought of as

$$f : A \to B \coprod \{\star\}$$

Example: $\mathrm{Par} : \mathbf{Set} \to \mathbf{Set}$ in **Cat**

Giving Par a restriction monad structure: Set $E = D = \mathrm{Par}$.

(R.1): "$f = f\overline{f}$"

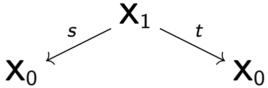
$$
\begin{array}{ccc}
\mathrm{Par} & \xrightarrow{\Delta} & \mathrm{Par}^2 \\
{\scriptstyle 1_{\mathrm{Par}}} \Big\| & & \Big\downarrow {\scriptstyle \mathrm{Par}\,\rho} \\
\mathrm{Par} & \xleftarrow[\mu.\mathrm{Par}\,\iota]{} & \mathrm{Par}^2
\end{array}
\quad \xmapsto{A} \quad
\begin{array}{ccc}
A\coprod\{\star\} & \xrightarrow{\Delta_A} & (A\coprod\{\star\})\coprod\{\star\} \\
{\scriptstyle 1_{\mathrm{Par}}(A)} \Big\| & & \Big\downarrow {\scriptstyle \mathrm{Par}(A)\,\rho_A} \\
A\coprod\{\star\} & \xleftarrow[\mu.\mathrm{Par}(A)\,\iota_A]{} & (A\coprod\{\star\})\coprod\{\star\}
\end{array}
$$

Implies that $\rho = 1$.

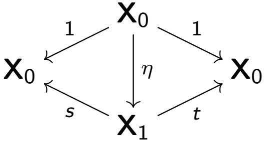
Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in **Span(Set)**

Let $\mathbf{X}$ be a small restriction category.

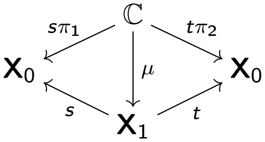
$T =$



$\eta : 1_T \Rightarrow T : \mathbf{X}_0 \to \mathbf{X}_1 : A \mapsto 1_A$

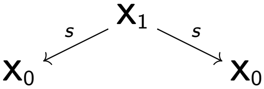


$\mu : T^2 \Rightarrow T : \mathbb{C} \to \mathbf{X}_1 : (f, g) \mapsto gf$

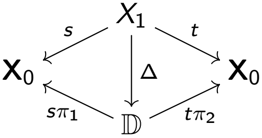


Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in **Span(Set)**

$D =$



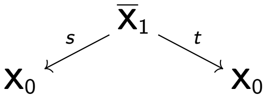
$$\Delta : T \Rightarrow TD : \mathbf{X}_1 \to \mathbb{D} : f \mapsto (f, f)$$



$$\mathbb{D} = \{(f, g) \in \mathbf{X}_1 \times \mathbf{X}_1 : sf = sg\}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in **Span(Set)**

$E =$

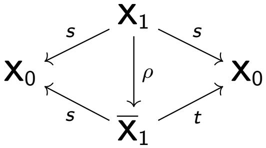


where

$$\overline{\mathbf{X}}_1 = \{\overline{f} : f \in \mathbf{X}\}$$

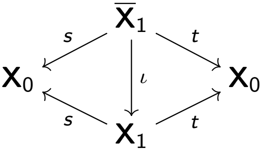
And define

$$\rho : D \Rightarrow E : \mathbf{X}_1 \to \overline{\mathbf{X}}_1 : f \mapsto \overline{f}$$

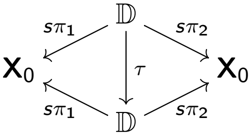


Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$\iota : E \Rightarrow T : \overline{\mathbf{X}}_1 \to \mathbf{X}_1 : \overline{f} \mapsto \overline{f}$$

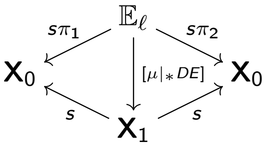


$$\tau : D^2 \Rightarrow D^2 : \mathbb{D} \to \mathbb{D} : (f, g) \mapsto (g, f)$$

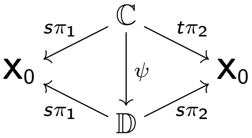


Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

$$[\mu\,|_*\,DE] : DE \Rightarrow D : \mathbb{E}_\ell \to \mathbf{X}_1 : (\overline{f}, g) \mapsto g\overline{f}$$



$$\psi : DT \Rightarrow TD : \mathbb{C} \to \mathbb{D} : (f, g) \mapsto (gf, f)$$



Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

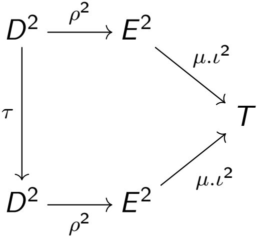
(R.1): "$f = f\overline{f}$"

$$\begin{array}{ccc} T & \xrightarrow{\ \Delta\ } & TD \\ {\scriptstyle 1_T}\Big\| & & \Big\downarrow{\scriptstyle T\rho} \\ T & \xleftarrow[\ \mu.T\iota\ ]{} & TE \end{array}$$

$$f \mapsto (f, f) \mapsto (\overline{f}, f) \mapsto f\,\overline{f}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathbf{Span(Set)}$

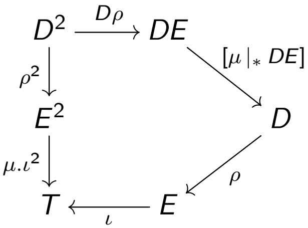
(R.2): "$\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$"



$$(f, g) \mapsto (\overline{f}, \overline{g}) \mapsto \overline{g}\,\overline{f}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

(R.3): “$\overline{g\overline{f}} = \overline{g}\,\overline{f}$”



$$(f, g) \mapsto (\overline{f}, g) \mapsto g\overline{f} \mapsto \overline{g\overline{f}}$$

Example: $R(\mathbf{X}) : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathbf{Span}(\mathbf{Set})$

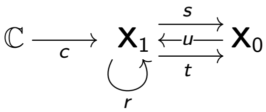
(R.4): "$\overline{g}f = f\overline{gf}$"

$$\begin{array}{ccccc} DT & \xrightarrow{\rho T} & ET & \xrightarrow{\mu.\iota T} & T \\ {\scriptstyle \psi}\downarrow & & & & \uparrow{\scriptstyle \mu.T\iota} \\ TD & & \xrightarrow[T\rho]{\qquad\qquad} & & TE \end{array}$$

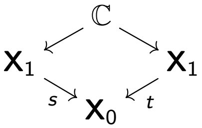
$$(f,g) \mapsto (gf, f) \mapsto (\overline{gf}, f) \mapsto f\,\overline{gf}$$

Restriction Category Objects

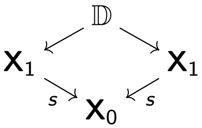
A restriction category in **C** (a category with pullbacks over s and t) contains the following data:



$\mathbb{C}$ and $\mathbb{D}$ are defined by the pullback squares



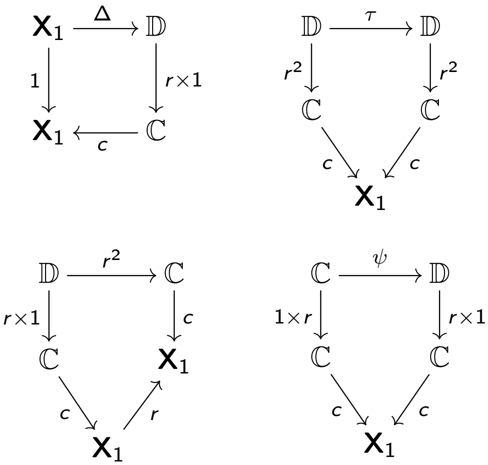
and



Satisfying the usual category axioms and

$$sr = s = tr$$

Restriction Category Objects: (R.1) – (R.4)



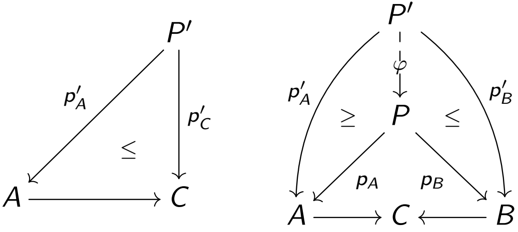
Restriction Category Objects

Definition

A double restriction category is a restriction category internal to **rCat**.

Restricted Pullbacks

Given any cospan $A \longrightarrow C \longleftarrow B$, a restricted pullback is cone consisting of an object P and total arrows $p_{A,B,C} : P \to A/B/C$ satisfying the following universal property:
For each lax cone (P', p'_A, p'_B, p'_C) over $A \longrightarrow B \longleftarrow C$, there is a unique $\varphi : P' \to P$ such that $\varphi \circ p \leq p'$ and $\overline{\varphi} = \overline{p'_A}\,\overline{p'_B}\,\overline{p'_C}$



Let **X** be a restriction category. A collection $\mathcal{M}$ of monics in **X** is *stable under restricted pullbacks* whenever:

- $\mathcal{M}$ contains all isomorphisms of $\mathcal{M}$,
- $\mathcal{M}$ is closed under composition,
- for each $m : B \to C$ in $\mathcal{M}$ and $f : A \to C$ in **X**, the restricted pullback

$$\begin{array}{ccc} A \otimes_C B & \xrightarrow{p_2} & B \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle m} \\ A & \xrightarrow[f]{} & C \end{array}$$

  of m along f exists and $p_1 \in \mathcal{M}$.

Define a restriction category $\mathrm{Par}(\mathbf{X}, \mathcal{M})$ (Cockett and Lack, 2002) with the following data:

- Objects: Same objects as $\mathbf{X}$
- Arrows: Isomorphism classes of spans

$$X \xleftarrow{\;i\;} D \xrightarrow{\;f\;} Y \;,$$

  with $i \in \mathcal{M}$.
  - Composition: restricted pullback
  - Restriction: $\overline{(i, f)} = (i, i)$

Example: Double Category $\mathbb{P}\mathrm{ar}(\mathbf{X}, \mathcal{M})$

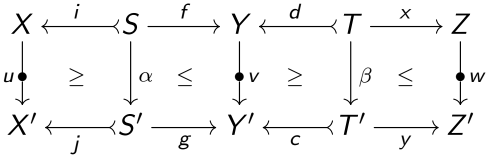
- Objects: Same as $\mathbf{X}$
- Vertical arrows: The total arrows of $\mathbf{X}$
  - total maps form a subcategory so composition is inherited from $\mathbf{X}$.
- Horizontal arrows: the arrows of $\mathrm{Par}(\mathbf{X}, \mathcal{M})$
  - composition by restricted pullbacks
- Double cells:

$$
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
{\scriptstyle u}\downarrow & \geq & \downarrow{\scriptstyle \alpha} & \leq & \downarrow{\scriptstyle v} \\
X' & \xleftarrow[\;i'\;]{} & D' & \xrightarrow[\;f'\;]{} & Y'
\end{array}
$$

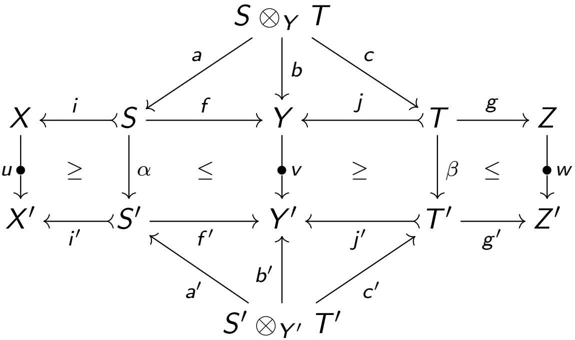
Double Cell Composition

Vertical Composition : compose all arrows vertically – straightforward

Horizontal Composition: given by universal property of restricted pullback



First take the restricted pulbacks:



This gives a lax cone

$$\begin{array}{ccccc} & & S \otimes_Y T & & \\ & \swarrow^{\alpha a} & & \searrow^{\beta c} & \\ S' & \leq & \downarrow{\scriptstyle vb} & \geq & T' \\ & \searrow_{f'} & & \swarrow_{j'} & \\ & & Y' & & \end{array}$$

over

$$S' \xrightarrow{f'} Y' \xleftarrow{j'} T'$$

So there is a unique a unique $\varphi : S \otimes_Y T \to S' \otimes_{Y'} T'$ giving the double cell

$$\begin{array}{ccccc}
X & \xleftarrow{\;ia\;} & S \otimes_Y T & \xrightarrow{\;gc\;} & Z \\
{\scriptstyle u}\downarrow & \geq & \downarrow{\scriptstyle \varphi} & \leq & \downarrow{\scriptstyle w} \\
X' & \xleftarrow[\;i'a'\;]{} & S' \otimes_{Y'} T' & \xrightarrow[\;g'c'\;]{} & Z'
\end{array}$$

Vertical Restriction

For each such α, define the vertical restriction $\widetilde{\alpha}$ of α to be

$$\widetilde{\alpha} = \begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\prec} & D & \overset{f}{\longrightarrow} & Y \\ {\scriptstyle \overline{u}=1_X}\Big\| & \geq & \Big\downarrow{\scriptstyle \overline{\alpha}} & \leq & \Big\|{\scriptstyle 1_Y=\overline{v}} \\ X & \underset{i}{\longleftarrow\!\!\prec} & D & \underset{f}{\longrightarrow} & Y \end{array}$$

Horizontal Restriction

For each such α, define the horizontal restriction $\overline{\alpha}$ of α to be

$$\overline{\alpha} = \begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\prec} & D & \overset{i}{\succ\!\!\longrightarrow} & X \\ u \big\downarrow\!\bullet & \geq & \big\downarrow \alpha & \leq & \bullet\!\big\downarrow u \\ X' & \underset{j}{\longleftarrow\!\!\prec} & D' & \underset{j}{\succ\!\!\longrightarrow} & X' \end{array}$$

It is quickly seen that the restriction structures commute:

$$
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
{\scriptstyle u}\,\downarrow\!\bullet & \geq & \downarrow{\scriptstyle \alpha} & \leq & \downarrow\!\bullet\,{\scriptstyle v} \\
X' & \xleftarrow[\;i'\;]{} & D' & \xrightarrow[\;f'\;]{} & Y'
\end{array}
\xrightarrow{\widetilde{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
\| & \geq & \downarrow{\scriptstyle \overline{\alpha}} & \leq & \| \\
X & \xleftarrow[\;i\;]{} & D & \xrightarrow[\;f\;]{} & Y
\end{array}
\xrightarrow{\overline{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;i\;} & X \\
\| & \geq & \downarrow{\scriptstyle \overline{\alpha}} & \leq & \| \\
X & \xleftarrow[\;i\;]{} & D & \xrightarrow[\;i\;]{} & X
\end{array}
$$

$$
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;f\;} & Y \\
{\scriptstyle u}\,\downarrow\!\bullet & \geq & \downarrow{\scriptstyle \alpha} & \leq & \downarrow\!\bullet\,{\scriptstyle v} \\
X' & \xleftarrow[\;i'\;]{} & D' & \xrightarrow[\;f'\;]{} & Y'
\end{array}
\xrightarrow{\overline{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;i\;} & X \\
{\scriptstyle u}\,\downarrow\!\bullet & \geq & \downarrow{\scriptstyle \alpha} & \leq & \downarrow\!\bullet\,{\scriptstyle u} \\
X' & \xleftarrow[\;j\;]{} & D' & \xrightarrow[\;j\;]{} & X'
\end{array}
\xrightarrow{\widetilde{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{\;i\;} & D & \xrightarrow{\;i\;} & X \\
\| & \geq & \downarrow{\scriptstyle \overline{\alpha}} & \leq & \| \\
X & \xleftarrow[\;i\;]{} & D & \xrightarrow[\;i\;]{} & X
\end{array}
$$