THE EHRESMANN-SCHEIN-NAMBOORIPAD THEOREM FOR INVERSE CATEGORIES

DARIEN DEWOLF AND DORETTE PRONK

Abstract. The Ehresmann-Schein-Nambooripad (ESN) Theorem asserts an equivalence between the category of inverse semigroups and the category of inductive groupoids. In this paper, we consider the category of inverse categories and functors – a natural generalization of inverse semigroups and semigroup homomorphisms – and extend the ESN Theorem to an equivalence between this category and the category of locally complete inductive groupoids and locally inductive functors. From the proof of this extension, we also generalize the ESN Theorem to an equivalence between the category of inverse semicategories and the category of locally inductive groupoids and to an equivalence between the category of inverse categories with oplax functors and the category of locally complete inductive groupoids and ordered functors.

1. Introduction

The Ehresmann-Schein-Nambooripad (ESN) Theorem asserts the existence of an equivalence between the category of inverse semigroups (with semigroup homomorphisms) and the category of inductive groupoids (with inductive functors). A groupoid is called *ordered* in this context if there is a compatible (functorial) order on both objects and arrows with a notion of restriction on the arrows such that an arrow $f\colon A \to B$ has a unique restriction $f'\colon A' \to B'$ with $f' \leq f$ for any object $A' \leq A$. For the precise definition see Definition 2.2, but a category theorist may like to think of these as groupoids internal to the category of posets with some additional properties. *Ordered functors* between these are functors that preserve the order. Furthermore, an ordered groupoid is called *inductive* when the objects form a meet-semilattice that has all finite meets except possibly the empty one, i.e., the top element. Furthermore, an ordered functor is inductive when it preserves the meets. The correspondence of the ESN Theorem is directly extendable to inverse semigroups and prehomomorphisms when one takes ordered functors, rather than inductive functors, between the inductive groupoids.

This theorem has been extended to various larger classes of semigroups. Nambooripad has studied the case of regular semigroups extensively [11, 12, 13], where Gould has extended the correspondence to left restriction semigroups (originally known as weakly left E-ample semigroups) [4], and Lawson to two-sided restriction semigroups (also called Ehresmann semigroups) [8]. Hollings extended this work to more general restriction

Both authors thank NSERC for its funding of this research.

2010 Mathematics Subject Classification: 18B35, 18B40.

Key words and phrases: Inverse semigroup, inverse category, inductive groupoid, locally complete inductive groupoid, inverse semicategory.

groups [6] with either semigroup homomorphisms or $\wedge$- or $\vee$-prehomomorphisms. The main ideas in this context have focused on either changing the requirement for a meet-semilattice structure to a different order structure on the objects of the groupoid, or on generalizing to inductive categories rather than groupoids.

Our approach here is to generalize this equivalence in a different direction. Semigroups can be viewed as single-object semicategories and we want to obtain a 'multi-object' version of the correspondence. As groupoids can be thought of as the multi-object version of groups, we think of *inverse categories* as a multi-object version of inverse semigroups. In this paper, we prove a new generalization of the ESN theorem which extends the result to inverse categories. Since we are generalizing the concept of *inverse* semigroup, we will remain within the category of groupoids. They will still be ordered, but the order structure will only be *locally inductive* in a suitable sense: the objects need to form a disjoint union of meet-semilattices. Since inverse categories have identities, we further require that each meet-semilattice has a top-element. We will call such groupoids *locally complete inductive*. If we instead generalize to *inverse semicategories*, this requirement is not needed and the groupoids will be called *locally inductive*. *Locally inductive functors*, ordered functors that preserve all meets that exist, will correspond to functors of inverse semicategories (Corollary 3.18). We will also show that the category of inverse categories and oplax functors is equivalent to the category of locally complete inductive groupoids and locally inductive functors, generalizing the classical result that the category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors (Theorem 3.21).

This work can also be viewed as a contribution to the work modeling local symmetry in terms of various types of groupoids and inverse semigroups. Important contributions on the relation between étale groupoids, inverse semigroups and quantales have been made by Resende. For a good overview of these results, see his lecture notes [16]. We plan to revisit the correspondence given in these notes in more detail in a sequel to this work where we will consider join inverse categories and the connection with étendues. Another related notion is that of a restriction category, as introduced by Cockett and Lack [1]. Inverse categories come with a natural notion of restriction, since they can be viewed as categories where all arrows are partial isomorphisms. Although they have a more general role to play in the various correspondences between models for partial symmetry, we will use them in this paper primarily to motivate our constructions on inverse categories.

In private communication, Lawson provided an unpublished preprint in which he provides a similar construction to provide a proof of our main result. His constructions rely on the existence of (maximal) identities in an inverse category, while ours relies on the partitioning of the objects into meet-semilattices. The advantage of our approach is that the classical ESN Theorem is a direct corollary of the equivalence between locally complete inductive groupoids by simply removing identities and applying our construction to a single-object inverse semicategory.

The groupoid we construct for an inverse category was also independently considered in the work of Linckelmann [10] on category algebras. Linckelmann observes that this groupoid has the same category algebra as the original inverse category, giving the category algebra of an inverse category the structure of a groupoid algebra: a groupoid algebra over a commutative ring is a direct product of matrix rings. In this paper, we introduce this groupoid with an ordered structure and observe the important characterizing properties of the order structure to obtain an equivalence of categories between the category of inverse categories and the category of these locally complete inductive groupoids.

From the semigroup perspective, this raises the question of whether there are appropriate multi-object versions of the other classes of semigroups mentioned above which then may be shown to be equivalent to appropriate versions of locally inductive categories.

2. Background

2.1. Inductive groupoids and inverse semigroups. Inductive groupoids are a class of groupoids whose arrows are equipped with a partial order satisfying certain conditions and whose objects form a meet-semilattice. Charles Ehresmann [3] used ordered groupoids to model pseudogroups while inverse semigroups, introduced by Gordon Preston [15], were concurrently used as an alternate model for pseudogroups. Ehresmann was certainly aware of the connection between ordered (inductive) groupoids and inverse semigroups, as it was Ehresmann who first introduced the tensor product (which we will call the *star product* for reasons detailed below) required to make the correspondence work. Boris Schein [17] made this connection explicit, requiring that the set of objects form a meet-semilattice, thus guaranteeing the existence of this star product for all arrows of the groupoid. K.S.S. Nambooripad [11, 12, 13, 14] independently developed the theory of so-called regular systems and their correspondence to so-called regular groupoids. This theory is, in fact, more general and specializes to the correspondence of inverse semigroups to inductive groupoids. A more detailed history of inverse semigroups, inductive groupoids and their applications can be found in Hollings' [5]. In this section, we present the modern exposition of this correspondence, which can be found in Mark Lawson's book [9].

2.2. Definition. *A groupoid* $\mathbf{G}$ *is said to be an* ordered groupoid *whenever there is a partial order* $\leq$ *on its arrows (which extends to its objects by comparing their identity arrows) satisfying the following four conditions:*

(i) *For each arrow* $f, g \in G$, $f \leq g$ *implies* $f^{-1} \leq g^{-1}$.

(ii) *For all arrows* $f, f', g, g' \in G$ *such that* $f \leq f'$, $g \leq g'$ *and the composites* fg *and* $f'g'$ *exist,* $fg \leq f'g'$.

(iii) *For each arrow* $f : A' \to B$ *in* G *and each object* $A \leq A'$ *in* G, *there exists a unique* restriction of f to A, *denoted* $[f|_* A]$, *such that* $\mathrm{dom}[f|_* A] = A$ *and* $[f|_* A] \leq f$.

(iv) *For each arrow* $f : A \to B'$ *in* G *and objects* $B \leq B'$ *in* G, *there exists a unique* corestriction of f to B, *denoted* $[B\,{}_*|f]$, *such that* $\mathrm{cod}[B\,{}_*|f] = B$ *and* $[B\,{}_*|f] \leq f$.

An ordered groupoid is said to be an inductive groupoid *whenever its objects form a meet-semilattice.*

2.3. Remark. In Lawson's terminology, meet-semilattices only need to be closed under finite non-empty meets, i.e., they do not need to have a top element. To keep the terminology simple we will adopt this custom and make explicit mention of the tops when we do want them to be there.

Though it is sometimes convenient to explicitly give both the restrictions and corestrictions in an ordered groupoid, the following proposition makes it necessary only to include one of them in any proofs.

2.4. Proposition. [9] *In Definition 2.2, conditions (iii) and (iv) are equivalent.*

2.5. Definition. [2, 3, 9] *Let* $\mathbf{G}$ *be an ordered groupoid with arrows* $\alpha, \beta \in \mathbf{G}$. *If* $\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)$ *exists, the* star product $\alpha \star \beta$ *of* α *and* β *is defined as*

$$\alpha \star \beta = [\alpha\,|_*\, \mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)][\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)\,{}_*|\, \beta].$$

2.6. Note. *What we call the star product here was already considered by Ehresmann in [2], where he called it the* restricted composition*. In the semigroup literature it is often called the tensor product. We choose to use* $\star$ *instead of* $\otimes$ *because tensor products are usually reserved for coequalisers and are not typically strictly associative. The star product, however, is strictly associative.*

2.7. Proposition. [9] *Let* $\mathbf{G}$ *be an inductive groupoid. This star product is associative and admits pseudoinverses given by the inverses in the inductive groupoid, making* $(\mathbf{G}_1, \star)$ *an inverse semigroup.*

Proof Sketch. For any pair of arrows in $\mathbf{G}$, one can show that the set

$$\langle \alpha, \beta \rangle = \{(\alpha', \beta') |\, \mathrm{cod}(\alpha') = \mathrm{dom}(\beta'), \alpha' \leq \alpha, \beta' \leq \beta\}$$

contains a unique maximal element (α', β') with $\alpha \star \beta = \beta' \circ \alpha'$. Since defined by composition, this star product is therefore associative. ■

2.8. Proposition. *For all objects* $A \leq B$ *of an ordered groupoid,* $[1_B\,|_*\, A] = 1_A = [A\,{}_*|\, 1_B]$.

Proof. The partial order on arrows induces the partial order on the objects of an ordered groupoid. Since an object of a category is identified by the identity arrow on that object, we have that $1_A \leq 1_B$. Since the (co)domain of 1_A is A, we have that $[1_B\,|_*\, A] = 1_A = [A\,{}_*|\, 1_B]$ by the uniqueness of (co)restrictions ■

2.9. Definition. *A morphism* $F : \mathbf{G} \to \mathbf{H}$ *of ordered groupoids (an* ordered functor*) is a functor such that, for all arrows* $f \leq g$ *in* $\mathbf{G}$, $F(f) \leq F(g)$ *in* $\mathbf{H}$. *An ordered functor between inductive groupoids is said to be* inductive *whenever it preserves the meet structure on objects.*

2.10. Notation. We denote the category of ordered groupoids and ordered functors by **oGrpd** and the category of inductive groupoids and inductive functors by **iGrpd**.

We will now briefly review Lawson's description of functorial constructions that form the equivalence of categories between the category of inverse semigroups and the category of inductive groupoids. We remind the reader that full details can be found in [9].

2.11. Construction. [Inverse Semigroups to Inductive Groupoids] *Given an inverse semigroup* $(S, \bullet)$, *define an inductive groupoid* $\mathcal{G}(S)$ *with the following data:*

- *Objects:* $\mathcal{G}(S)_0 = E(S)$, *the idempotents in* S. *Since* S *is an inverse semigroup,* $E(S)$ *is a meet-semilattice with meets given by the product in* S. *We note here that this meet-semilattice of idempotents will be complete in our sense (have a top element) if and only if* S *is an inverse monoid (has a unit element).*

- *Arrows: For each element* $s \in S$, *there is an arrow* $s : s^\bullet s \to ss^\bullet$ *(we remind the reader that* $s^\bullet$ *denotes the partial inverse of* s*). Composition is given by multiplication in* S *and identities are the elements of* $E(S)$.

- *Inverses: For each arrow* $s : s^\bullet s \to ss^\bullet$ *in* $\mathcal{G}(S)$, *define* $s^{-1} = s^\bullet$, *its pseudoinverse in* S.

- *The partial order on arrows is given by the natural partial order (*$s \leq t$ *if and only if* $s = te$ *for some idempotent* e*) on the elements of* S. *It can be checked that this partial order satisfies conditions (i) and (ii) of an ordered groupoid.*

- *The (co)restrictions are also given by multiplication in* S. *This can be checked to satisfy condition (iii) of an ordered groupoid.*

2.12. Construction. [Inductive Groupoids to Inverse Semigroups] *Given an inductive groupoid* $(\mathbf{G}, \leq)$, *define an inverse semigroup* $\mathcal{S}(\mathbf{G})$ *whose elements are the arrows of* $\mathbf{G}$ *and whose multiplication is given by the star product. This is an inverse semigroup operation with inverses those from* $\mathbf{G}$ *(Proposition 2.7).*

2.13. Theorem. [ESN, [9]] *The constructions* $\mathcal{G}$ *and* $\mathcal{S}$ *are functorial and form an equivalence of categories*

$$\mathbf{iGrpd} \underset{\mathcal{G}}{\overset{\mathcal{S}}{\rightleftarrows}} \mathbf{iSgp}$$

2.14. Inverse categories and restriction categories. *Restriction categories*, as defined by Cockett and Lack [1], are categories equipped with enough structure to encode "partiality" of morphisms. In particular, each morphism f in a restriction category is assigned a *restriction idempotent* $\overline{f}$. This assignment satisfies some axioms which algebraically encode the intuition of $\overline{f}$ playing the role of "the domain on which the partial morphism f is defined", which facilitates computation in categories with some natural notion of partial morphism.

Given that inverse semigroups model partial automorphisms, restriction categories will provide a natural setting in which to study their multi-object analogue, inverse categories.

This subsection reviews some basic restriction category theory, which will be used in the motivation and proof of our main result.

2.15. Definition. [1] *A restriction structure on a category* $\mathbf{X}$ *is an assignment of an arrow* $\overline{f} : A \to A$ *to each arrow* $f : A \to B$ *in* $\mathbf{X}$ *satisfying the following four conditions:*

(R.1) For all maps f, $f\,\overline{f} = f$.

(R.2) For all maps $f : A \to B$ *and* $g : A \to B'$, $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

(R.3) For all maps $f : A \to B$ *and* $g : A \to B'$, $\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$.

(R.4) For all maps $f : B \to A$ *and* $g : A \to B'$, $\overline{g}\,f = f\,\overline{gf}$.

A category equipped with a restriction structure is called a restriction category.

2.16. Definition. [1] *A* restriction functor $F : \mathbf{X} \to \mathbf{Y}$ *between restriction categories is a functor which preserves the restriction idempotents;* $F\left(\overline{f}\right) = \overline{F(f)}$ *for all* $f \in \mathbf{X}_1$.

2.17. Example. Examples of restriction categories:

(a) **Par** as defined above is the prototypical example of a restriction category. The axioms (R.1) – (R.4) required for **Par** to be a restriction category are easily verified. We can interpret expressions such as $f\,\overline{g}$ as "f restricted to where g is defined".

(b) Let $\mathbf{C}$ be an ordinary category equipped with a stable system $\mathcal{M}$ of monics (all details of this example can be found in [1]). Define a category $\mathrm{Par}(\mathbf{C}, \mathcal{M})$ with the following data:

- Objects: Same objects as $\mathbf{C}$.
- Arrows: Isomorphism classes of spans

$$X \xleftarrow{\;i\;} D \xrightarrow{\;f\;} Y,$$

  where $i \in \mathcal{M}$. We will sometimes denote such an arrow (actually, its isomorphism class) as (i, f).
- Composition: Composition is given by pullback.

- Restrictions: Given any arrow (i, f), the assignment $\overline{(i, f)} = (i, i)$ defines a restriction structure on $\mathrm{Par}(\mathbf{C}, \mathcal{M})$.

The next lemma lists some useful identities that will be used, without reference, to make calculations in restriction categories.

2.18. Lemma. [1] *If* $\mathbf{X}$ *is a restriction category, then:*

(i) $\overline{f}$ *is idempotent;*

(ii) $\overline{f}\,\overline{gf} = \overline{gf}$;

(iii) $\overline{\overline{g}f} = \overline{gf}$;

(iv) $\overline{\overline{f}} = \overline{f}$;

(v) $\overline{\overline{g}\,\overline{f}} = \overline{g}\,\overline{f}$;

(vi) if f *is monic, then* $\overline{f} = 1$;

(vii) $f\overline{g} = f$ *implies* $\overline{f} = \overline{f}\,\overline{g}$.

2.19. Note. *A restriction category* $\mathbf{X}$ *has a natural, locally partially ordered 2-category structure: for any two parallel arrows* $f, g : C \to D$ *in* $\mathbf{X}$, *we define a partial order by* $f \leq g$ *if and only if* $f = g\overline{f}$. *Notice that if* $f \leq g$, *then*

$$\overline{g}\,\overline{\overline{f}} = \overline{g}\,\overline{f} = \overline{g\overline{f}} = \overline{f}$$

and thus $\overline{f} \leq \overline{g}$.

2.20. Proposition. *Suppose that* f, f', g *and* g' *are arrows in a restriction category* $\mathbf{X}$ *with* $f \leq f'$ *and* $g \leq g'$. *If the composites* fg *and* $f'g'$ *exist, then* $fg \leq f'g'$.

Proof. Suppose that f, f', g and g' are arrows in $\mathbf{X}$ with $f \leq f'$ and $g \leq g'$, and such that the composites fg and $f'g'$ exist. Then

$$f'g'\overline{fg} = f'g'\overline{g}\,\overline{fg} = f'g\overline{fg} = f'\overline{f}g = fg$$

and thus $fg \leq f'g'$. ∎

2.21. Definition. *A map f in a restriction category* $\mathbf{X}$ *is called* total *whenever* $\overline{f} = 1$.

2.22. Lemma. [1] *If* $\mathbf{X}$ *is a restriction category, then:*

(i) every monomorphism is total;

(ii) if f and g are total, then gf is total;

(iii) if gf is total, then f is total;

(iv) the total maps form a subcategory, denoted $\mathrm{Tot}(\mathbf{X})$.

2.23. Definition. *A morphism* $F : \mathbf{X} \to \mathbf{Y}$ *of restriction categories (a* restriction functor*) is a functor such that* $F\left(\overline{f}\right) = \overline{F(f)}$ *for each* $f \in X_1$.

Inverse categories. As groupoids are for groups, we will use a structure describing multi-object inverse semigroups. Inverse semigroups with units are exactly single-object inverse categories, so it seems that inverse (semi)categories could be appropriate for such a role.

2.24. Definition. [7] *A category* $\mathbf{X}$ *is said to be an* inverse category *whenever, for each arrow* $f : A \to B$ *in* $\mathbf{X}$, *there exists a unique* $f^\circ : B \to A$ *in* $\mathbf{X}$ *such that* $f \circ f^\circ \circ f = f$ *and* $f^\circ \circ f \circ f^\circ = f^\circ$.

2.25. Definition. *A map f in a restriction category* $\mathbf{X}$ *is called a* restricted isomorphism *whenever there exists a map g – called a* restricted inverse *of f – such that* $gf = \overline{f}$ *and* $fg = \overline{g}$.

Following from the commutation of idempotents (Restriction Category Axiom (R.2) in Definition 2.15) we have the following property of restricted isomorphisms:

2.26. Theorem. [Lemma 2.18(vii), [1]] *If f is a restricted isomorphism, then its restricted inverse is necessarily unique.*

2.27. Note. *If a category* $\mathbf{X}$ *has the* property *of being an inverse category, one can define a restriction* structure *on* $\mathbf{X}$ *by defining* $\overline{f} = f^\circ f$. *Indeed, with this restriction structure, every arrow in* $\mathbf{X}$ *is a restricted isomorphism and the restricted inverse of an arrow f is exactly f°. This justifies the following notation and definition.*

2.28. Notation. Given a map f in a restriction category $\mathbf{X}$, we denote its restricted inverse (if it exists) by f°.

2.29. Definition. *A restriction category* $\mathbf{X}$ *is called an* inverse category, *whenever every map f is a restricted isomorphism.*

2.30. Example. Some inverse categories:

(a) The category of sets and partial bijections.

(b) Any inverse semigroup with unit is a single-object inverse category.

(c) Any groupoid is an inverse category with all arrows total.

2.31. LEMMA. [1] *If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, then F preserves*

(i) total maps,

(ii) restriction idempotents,

(iii) restricted sections and

(iv) restricted isomorphisms.

2.32. NOTE. *Any functor between inverse categories is a restriction functor preserving restricted isomorphisms. This follows from the restriction structure and restricted isomorphisms being defined as specific composites. We will therefore omit the words "inverse" and "restriction" when speaking of functors between inverse categories.*

As expected, restriction idempotents are their own restricted inverse.

2.33. PROPOSITION. *In an inverse category, $\left(\overline{f}\right)^\circ = \overline{f}$ for all arrows f.*

PROOF. Since all arrows in an inverse category are restricted isomorphisms,

$$\left(\overline{f}\right)^\circ = (f^\circ f)^\circ = f^\circ (f^\circ)^\circ = f^\circ f = \overline{f}.$$

■

It is clear that inverse categories, interpreted as restriction categories in Definition 2.29, are exactly the same as inverse categories interpreted as multi-object inverse semigroups in Definition 2.24; that is restrictions come for free in an inverse category and are given by $\overline{f} = f^\circ f$. In this paper, we choose to think in terms of restriction categories for two reasons: firstly, the choice of notation in restriction categories facilitates calculations. Secondly, we prefer to think of (finite) inverse semigroups as collections of partial automorphisms on a (finite) set whose idempotents are partial identities – inverse categories in terms of restriction categories explicitly make use of this intuition.

2.34. NOTATION. We denote the category of inverse categories and functors by **iCat**.

3. Main result

In this section, we introduce the notion of locally complete inductive groupoids: ordered groupoids whose objects may be partitioned into meet-semilattices, each of which contain a top element. We will then give functorial constructions of locally complete inductive groupoids from inverse categories, and vice versa. These constructions will then be seen to give an equivalence of categories between **iCat** and **lciGrpd** (the category of locally complete inductive groupoids). The identities of an inverse category are seen to correspond to the tops of the meet-semilattices in a locally complete inductive groupoid and the equivalence can thus be immediately generalized to an equivalence between the category of inverse semicategories and semifunctors and the category of locally inductive groupoids and locally inductive functors. Finally, we end this section with a short discussion of a categorical analogue of the classical result in semigroup theory that the category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors. Explicitly, we show that the category of inverse categories and oplax functors is equivalent to the category of locally complete inductive groupoids and ordered functors.

3.1. Definition. *Let A be an object of a restriction category $\mathbf{X}$. Let E_A denote the set of restrictions of all endomorphisms on A. That is,*

$$E_A = \left\{ \overline{f} : A \to A | f : A \to A \in \mathbf{X} \right\}.$$

Notice that, for any $f : A \to B$ in $\mathbf{X}$, we have $(\overline{f} : A \to A) \in E_A$, since $\overline{\overline{f}} = \overline{f}$. The reason for specifying that the restrictions in E_A come from endomorphisms in $\mathbf{X}$, then serves no use further than simply reminding us that the equivalence we are trying to establish here is based on the observation that an inverse category is, at each object, an inverse semigroup (with identity).

3.2. Proposition. *For each object A of a restriction category $\mathbf{X}$, E_A is a meet-semilattice with meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$. In addition, E_A has top element 1_A.*

Proof. First of all, E_A is a poset with the natural partial order inherited from $\mathbf{X}$. We now show that E_A has finite meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$:

- First, it is a lower bound:

$$\overline{a}\,\overline{\overline{a} \wedge \overline{b}} = \overline{a}\,\overline{\overline{a}\overline{b}} = \overline{a}\,\overline{a}\overline{b} = \overline{a}\,\overline{b} = \overline{a} \wedge \overline{b}$$

  and thus $\overline{a} \wedge \overline{b} \leq \overline{a}$. Similarly, $\overline{a} \wedge \overline{b} \leq \overline{b}$.

- This lower bound is unique up to isomorphism (equality): suppose that $\overline{d}$ is such that $\overline{d} \leq \overline{a}$, $\overline{d} \leq \overline{b}$ and $\overline{a} \wedge \overline{b} \leq \overline{d}$. Then

$$\overline{d} = \overline{a}\,\overline{d} = \overline{a}\,\overline{b}\,\overline{d} = \overline{d}\,\overline{\overline{a}\,\overline{b}} = \overline{d}\,\overline{\overline{a} \wedge \overline{b}} = \overline{a} \wedge \overline{b}.$$

Finally, since $\overline{1_A} = 1_A$, $1_A \in E_A$. Also, given any $\overline{f} : A \to A$, $1_A\overline{f} = \overline{f}$ and thus $\overline{f} \leq 1_A$ and 1_A is the top element of E_A. ■

We may now give the (functorial) constructions giving an equivalence between the category of locally complete inductive groupoids and inverse categories.

3.3. Construction. *Given an inverse category $\left(\mathbf{X}, \circ, \overline{(-)}\right)$, define a groupoid $(\mathcal{G}(\mathbf{X}), \bullet, \leq)$ with the following data:*

- *Objects:* $\mathcal{G}(\mathbf{X})_0 = \coprod_{A \in \mathbf{X_0}} E_A$.

- *Arrows: Every arrow in $\mathcal{G}(\mathbf{X})$ is of the form $f : \overline{f_A} \to \overline{f^\circ_B}$ for each arrow $f : A \to B$ in $\mathbf{X}$.*
  - *Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$. This composite is indeed an arrow, for*
$$\overline{gf} = \overline{\overline{g}f} = \overline{\overline{f^\circ}f} = \overline{f}$$
and
$$\overline{(gf)^\circ} = \overline{f^\circ g^\circ} = \overline{\overline{f^\circ}g^\circ} = \overline{\overline{g}g^\circ} = \overline{g^\circ}.$$
  - *Identities: For any object $\overline{f} : A \to A$ in $\mathcal{G}(\mathbf{X})$, define $1_{\overline{f}} = \overline{f}$ (which is well-defined since $\overline{\overline{f}} = \overline{f}$). The identity then satisfies the appropriate axiom: for each $g : \overline{g} \to \overline{g^\circ}$ with $\overline{g} = \overline{f}$ and $\overline{g^\circ} = \overline{f^\circ}$, we have $\overline{f^\circ}g = \overline{g^\circ}g = g$ and $g\overline{f} = g\overline{g} = g$.*
  - *Inverses: Given an arrow $f : \overline{f} \to \overline{f^\circ}$, define $f^{-1} : \overline{f^\circ} \to \overline{f}$ to be f°, the unique restricted inverse of f from $\mathbf{X}$'s inverse structure. The composites are $ff^\circ = \overline{f^\circ} = 1_{\overline{f^\circ}}$ and $f^\circ f = \overline{f} = 1_{\overline{f}}$ as required.*

3.4. Definition. *An ordered groupoid is said to be a* locally inductive groupoid *whenever there is a partition $\{M_i\}_{i\in I}$ of $\mathbf{G}_0$ into meet-semilattices M_i with the property that any two comparable objects be in the same meet-semilattice M_i. A locally inductive groupoid is said to be* locally complete *whenever each meet-semilattice M_i admits a top-element $\top_i$.*

3.5. Note. *The requirement that any two comparable objects of a locally inductive groupoid be in the same meet-semilattice corresponds to our intuition that if the meet $A \wedge B$ of two objects A and B exists in M_i, then A and B, both sitting above this meet, should also be elements of M_i.*

3.6. Definition. *An ordered functor between locally inductive groupoids is said to be* locally inductive *whenever it preserves all meets that exist. In particular, a locally inductive functor will preserve empty meets and thus top elements and there is no requirement to define so-called "locally complete inductive functors".*

3.7. Notation. We denote the category of locally inductive groupoids and locally inductive functors by **liGrpd** and the category of locally complete inductive groupoids and locally inductive functors by **lciGrpd**.

3.8. Proposition. *For each inverse category $\mathbf{X}$, $\mathcal{G}(\mathbf{X})$ is a locally complete inductive groupoid.*

Proof. Recall that the partial order on the objects $\overline{f}$ in $\mathcal{G}(\mathbf{X})$ is that which is induced by the partial order on the arrows of $\mathbf{X}$. That is, $\overline{f} \leq \overline{g}$ if and only if $\overline{f} = \overline{g}\overline{\overline{f}} = \overline{g}\overline{f}$. We now prove that this partial order gives $\mathcal{G}(\mathbf{X})$ the structure of an ordered groupoid:

(i) Suppose that f and g are arrows in $\mathcal{G}(\mathbf{X})$ with $f \leq g$. That is, we suppose that $g\overline{f} = f$ (since these are also arrows in $\mathbf{X}$). Then

$$\begin{aligned} f^\circ &= (g\overline{f})^\circ = \overline{f}^\circ g^\circ = \overline{f} g^\circ = \overline{f} g^\circ g g^\circ = g^\circ g \overline{f} g^\circ \\ &= g^\circ g\overline{f}\,\overline{f} g^\circ = g^\circ g \overline{f}\,\overline{f}^\circ g^\circ = g^\circ g\overline{f}(g\overline{f})^\circ = g^\circ f f^\circ \\ &= g^\circ \overline{f^\circ} \end{aligned}$$

and thus $f^{-1} = f^\circ \leq g^\circ = g^{-1}$.

(ii) This follows directly from Proposition 2.20.

(iii) Given an arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ with an object $\overline{e} \leq \overline{\alpha}$, we define the restriction $[\alpha|_*\overline{e}]$ of α to $\overline{e}$ to be $\alpha\overline{e}$. This is indeed an arrow whose domain is $\overline{e} : \overline{\alpha\overline{e}} = \overline{\alpha}\,\overline{e} = \overline{e}$.

Also, $\alpha\overline{\alpha\overline{e}} = \alpha\overline{\alpha}\,\overline{e} = \alpha\overline{e}$, so that $\alpha\overline{e} \leq \alpha$.

If $\beta \leq \alpha$ is any other arrow with $\mathrm{dom}(\beta) = \overline{e}$, we have $\alpha\overline{\beta} = \beta$ and $\overline{\beta} = \overline{e}$, so that $\beta = \alpha\overline{e}$ and thus $[\alpha|_*\overline{e}]$ as defined is unique.

(iv) Given an arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ with an object $\overline{e} \leq \overline{\alpha^\circ}$, we define the corestriction $[\overline{e}\,{}_*|\alpha]$ of α to $\overline{e}$ to be $\overline{e}\alpha$. This is indeed an arrow whose codomain is $\overline{e} : \overline{(\overline{e}\alpha)^\circ} = \overline{\alpha^\circ\overline{e}} = \overline{a^\circ}\,\overline{e} = \overline{e}$.

Also, $\alpha\overline{\overline{e}\alpha} = \alpha\overline{e\alpha} = \alpha(e\alpha)^\circ e\alpha = \alpha\alpha^\circ e^\circ e\alpha = e^\circ e\alpha\alpha^\circ\alpha = \overline{e}\alpha$, so that $\overline{e}\alpha \leq \alpha$.

If $\beta \leq \alpha$ is any other arrow with $\mathrm{cod}(\beta) = \overline{e}$, we have $\beta^\circ \leq \alpha^\circ$ (property (i) of ordered groupoids) and thus $\alpha^\circ\overline{\beta^\circ} = \beta^\circ$ and $\overline{\beta^\circ} = \overline{e}$, so that $\beta^\circ = \alpha^\circ\overline{e} = (\overline{e}\alpha)^\circ$ and thus $[\overline{e}\,{}_*|\alpha]$ as defined is unique.

Given the choice of objects for $\mathcal{G}(\mathbf{X})$, it follows immediately from Propositions 3.2 and the fact that the E_As are disjoint that $\mathcal{G}(\mathbf{X})$ is a locally complete inductive groupoid. ∎

The composition in $\mathcal{G}(\mathbf{X})$ of f and g exists exactly when $\overline{f} = \overline{g^\circ}$ and is defined by the composition in $\mathbf{X}$. The star product in $\mathcal{G}(\mathbf{X})$ is a natural extension of this composition in the sense that it exists whenever the meet $\overline{f} \wedge \overline{g^\circ}$ exists. This lemma shows that this extension is also defined by the composition in $\mathbf{X}$.

3.9. Lemma. *If $\mathbf{X}$ is an inverse category, then in $\mathcal{G}(\mathbf{X})$ the star products (when defined) are given by composition in $\mathbf{X}$:*

$$f \star g = fg.$$

Proof. Recall that, for any arrow f in $\mathbf{X}$, $\mathrm{dom}(f) = \overline{f}$ and $\mathrm{cod}(f) = \overline{f^\circ}$. Then

$$\begin{aligned} f \star g &= [f\,|_*\,\mathrm{dom}(f) \wedge \mathrm{cod}(g)]\,[\mathrm{dom}(f) \wedge \mathrm{cod}(g)\,{}_*|\,g] \\ &= \left[f\,|_*\,\overline{f} \wedge \overline{g^\circ}\right]\left[\overline{f} \wedge \overline{g^\circ}\,{}_*|\,g\right] \\ &= \left[f\,|_*\,\overline{f}\,\overline{g^\circ}\right]\left[\overline{f}\,\overline{g^\circ}\,{}_*|\,g\right] = f\overline{f}\,\overline{g^\circ}\,\overline{f}\,\overline{g^\circ}g = f\overline{f}\,\overline{g^\circ}g = fg \end{aligned}$$

∎

3.10. PROPOSITION. *Locally inductive functors preserve star products.*

PROOF. This follows immediately from the definition of a locally inductive functor and the fact that any ordered functor preserves restrictions and corestrictions [9, Proposition 4.1.2(1)]. ■

3.11. PROPOSITION. *For each functor $F : \mathbf{X} \to \mathbf{Y}$ between inverse categories, there exists a locally inductive functor $\mathcal{G}(F) : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{Y})$.*

PROOF. We claim that $F : \mathbf{X} \to \mathbf{Y}$ induces a locally inductive functor $\mathcal{G}(F)$ between the groupoids $\mathcal{G}(\mathbf{X})$ and $\mathcal{G}(\mathbf{Y})$. Since F is a functor of inverse categories, we have, for each $\overline{f}$ in $\mathbf{X}$, that $F\overline{f} = \overline{F(f)}$ is a restriction idempotent in $\mathbf{Y}$. We can then define, for any object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, $\mathcal{G}(F)(\overline{f}) = F\overline{f}$ and this is a well-defined object function.

Given an arrow $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$, we define

$$\mathcal{G}(F)(f) := \left[F(f) : F\left(\overline{f}\right) \to F\left(\overline{f^\circ}\right)\right] = \left[F(f) : \overline{F(f)} \to \overline{F(f^\circ)}\right].$$

We check that this is indeed an arrow in $\mathcal{G}(\mathbf{Y})$. Clearly, $F(f)$ has the correct domain. We check, then, that it has the correct codomain; that is, we verify that $\overline{(F(f))^\circ} = \overline{F(f^\circ)}$. By Lemma 2.31 (iv), $(F(f))^\circ = F(f^\circ)$. It follows, then, that $\overline{(F(f))^\circ} = \overline{F(f^\circ)}$ and thus F is well defined on arrows.

Since the objects of $\mathcal{G}(\mathbf{X})$ are specific arrows in $\mathbf{X}$ and the composition in $\mathcal{G}(\mathbf{X})$ is, when defined, given by composition in $\mathbf{X}$, the functoriality of $\mathcal{G}(F)$ follows from the functoriality of F.

We check now that F is an ordered functor. That is, we must check that F preserves partial orders. Suppose that $f \leq g$ are arrows in $\mathcal{G}(\mathbf{X})$. Then $g\overline{f} = f$ and thus

$$F(g)\overline{F(f)} = F(g)F(\overline{f}) = F(g\overline{f}) = F(f).$$

Therefore, $F(f) \leq F(g)$ in $\mathcal{G}(\mathbf{Y})$ and F is an ordered functor.

Finally, we verify that F is a locally inductive functor. If $\overline{a} \wedge \overline{b}$ exists in $\mathcal{G}(\mathbf{X})$, then $\overline{a}$ and $\overline{b}$ are endomorphisms on the same object and are thus composable and in the same meet-semilattice. Then, by the functoriality of F, $F\left(\overline{a} \wedge \overline{b}\right) = F\left(\overline{a}\,\overline{b}\right) = F(\overline{a})F\left(\overline{b}\right) = F(\overline{a}) \wedge F\left(\overline{b}\right)$. ■

3.12. COROLLARY. *Construction 3.3 is the object function of a fully faithful functor* $\mathcal{G} : \mathbf{iCat} \to \mathbf{lciGrpd}$.

PROOF. By the proof of Proposition 3.11, $\mathcal{G}$ is clearly a faithful functor.

Let $\mathbf{X}$ and $\mathbf{X}'$ be inverse categories and suppose that $F : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{X}')$ is a locally inductive functor. We seek, then, a functor $F' : \mathbf{X} \to \mathbf{X}'$ with $\mathcal{G}(F') = F$.

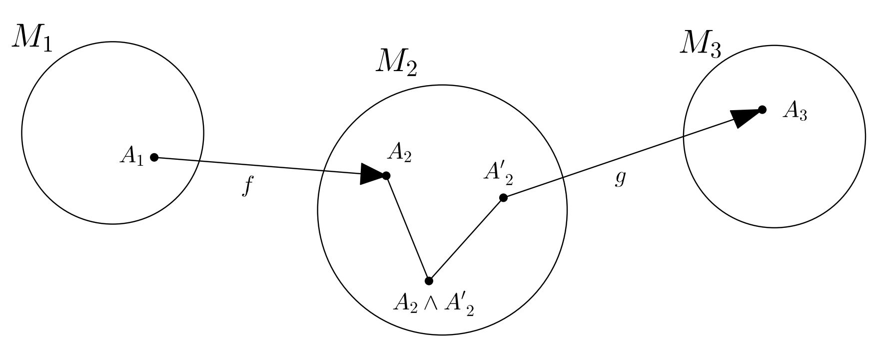
For any two restriction idempotents $\overline{e}$ and $\overline{f}$ in E_A, we have $F(\overline{e} \wedge \overline{f}) = F\overline{e} \wedge F\overline{f}$ since F is locally inductive. This implies that $F\overline{e}$ and $F\overline{f}$ are $\mathbf{X}'$-endomorphisms on the same object and thus $F(E_A) \subseteq E_B$ for some object $B \in \mathbf{X}'$. So we can define, for each object $A \in \mathbf{X}$, $F'(A)$ to be the object in $\mathbf{X}'$ satisfying $F(E_A) \subseteq E_{F'(A)}$ in $\mathcal{G}(\mathbf{X}')$.

Given any arrow $f : A \to B$ in $\mathbf{X}$, we must define an arrow $F'(f) : F'(A) \to F'(B)$ in $\mathbf{X}'$. We know that f corresponds to the arrow $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$, whose image under F is $F(f) : F\overline{f} \to F\overline{f^\circ}$ in $\mathcal{G}(\mathbf{X}')$. Since $F\overline{f} \in F(E_A)$ and $F\overline{f^\circ} \in F(E_B)$, this $F(f)$ corresponds to an arrow $F'(f) : F'(A) \to F'(B)$ in $\mathbf{X}'$.

Clearly, identity arrows in $\mathbf{X}$, corresponding to identity arrows in $\mathcal{G}(\mathbf{X})$ and mapped to identities in $\mathcal{G}(\mathbf{X}')$ under F, will be mapped to identities in $\mathbf{X}'$ under F'. We check that composition is preserved. Suppose that f and g are arrows whose composite gf exists in $\mathbf{X}$. Both g and f correspond, then, to arrows $g : \overline{g} \to \overline{g^\circ}$ and $f : \overline{f} \to \overline{f^\circ}$, respectively, in $\mathcal{G}(\mathbf{X})$. Notice that the composite gf does not necessarily exist (as an arrow) in $\mathcal{G}(\mathbf{X})$, but that, since $\overline{g}, \overline{f^\circ} \in E_B$, the product $g \star f$ does and that this star product uniquely corresponds to gf by Proposition 3.9. By Proposition 3.10 (since F preserves meets), then, $F(g \star f) = F(g) \star F(f)$ and, again by Lemma 3.9 and the definition of F', corresponds to $F'(g)F'(f)$. ■

3.13. Construction. *Given a locally complete inductive groupoid* $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i\in I})$, *define an inverse category* $\left(\mathcal{I}(\mathbf{G}), \circ, \overline{(-)}\right)$ *with the following data:*

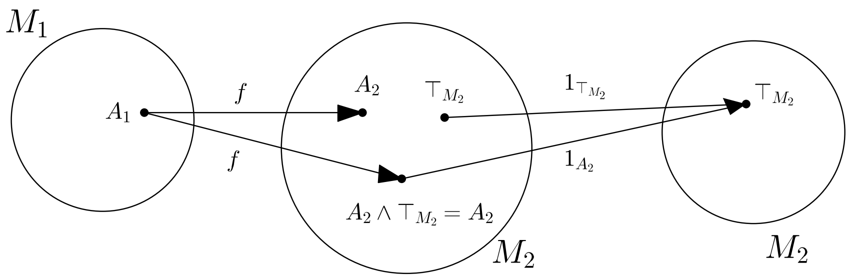
- *Objects: The objects are the meet-semilattices M_i.*
- *Arrows: $\mathcal{I}(\mathbf{G})(M_1, M_2) = \{f : A_1 \to A_2 \text{ in } \mathbf{G} \mid A_1 \in M_1,\ A_2 \in M_2\}$. Note that every object of $\mathbf{G}$ is in some M_i, and the M_i are disjoint, so that every arrow in $\mathbf{G}$ will be found in exactly one of these hom-sets.*
  - *Composition: A composable pair of arrows $f : M_1 \to M_2$ and $g : M_2 \to M_3$ in $\mathcal{I}(\mathbf{G})$, corresponds to a pair of arrows $f : A_1 \to A_2$ and $g : A_2' \to A_3$ in $\mathbf{G}$ with $A_1 \in M_1$, $A_2, A_2' \in M_2$ and $A_3 \in M_3$. Since M_2 is a meet-semilattice, the meet $A_2 \wedge A_2'$ exists. We can therefore define the composite of f with g as $g \circ f = g \star f = [g\,|_* A_2 \wedge A_2'][A_2 \wedge A_2'{}_*|\, f]$. This composition is associative by Proposition 2.7.*



– *Identities: For each object* M_1*, define* $1_{M_1} : M_1 \to M_1$ *to be* $1_{\top_{M_1}} = \top_{M_1} \to \top_{M_1}$ *in* $\mathbf{G}$*. Let* $f : M_1 \to M_2$ *be an arrow corresponding to* $f : A_1 \to A_2$ *in* $\mathbf{G}$*. Note that* $[1_{\top_{M_1}} |_* A_1 \wedge \top_{M_1}] = 1_{A_1}$ *by Proposition 2.8. Then*

$$f \circ 1_{\top_{M_1}} = [f |_* A_1 \wedge \top_{M_1}] \bullet \left[A_1 \wedge \top_{M_1} {}_*| 1_{\top_{M_1}}\right] = [f |_* A_1] \bullet 1_{A_1} = f.$$

Similarly, $1_{\top_{M_2}} \circ f = f$*.*



– *Restrictions: Given an arrow* $f : M_1 \to M_2$ *corresponding to an arrow* $f : A_1 \to A_2$ *in* $\mathbf{G}$*, define* $\overline{f} : M_1 \to M_1$ *by* $\overline{f} = 1_{A_1} : A_1 \to A_1$*. Conditions (R.1) – (R.4) saying that* $\mathcal{I}(\mathbf{G})$ *is a restriction category follow readily from the fact that all restriction idempotents are identities on some object in* $\mathbf{G}$ *and that restrictions in an ordered groupoid are unique.*

– *Partial Isomorphisms: For each arrow* $f : M_1 \to M_2$*, define* $f^\circ : M_2 \to M_1$ *as* $f^{-1} : A_2 \to A_1$*. To check that this is a restricted inverse, we check the required composites. First,*

$$f \circ f^\circ = f \star f^\circ = [f |_* A_1 \wedge A_1] \bullet [A_1 \wedge A_1 {}_*| f^{-1}] = f \bullet f^{-1} = 1_{A_2} = \overline{f^{-1}}.$$

Similarly, $f^\circ \circ f = \overline{f}$*.*

3.14. Proposition. *For each locally inductive functor* $F : \mathbf{G} \to \mathbf{H}$*, there exists a functor* $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{H})$*.*

Proof. We show that F induces a functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{H})$.

Given any object in $\mathcal{I}(\mathbf{G})$, a meet-semilattice M_1, define $\mathcal{I}(F)(M_1)$ to be *the* meet-semilattice M_1' such that $F(M_1) \subseteq M_1'$. Note that this assignment of M_1' to M_1 is unique since the M_i' are a partition of $\mathbf{H}_0$.

For any arrow $f : M_1 \to M_2$ in $\mathcal{I}(\mathbf{G})$ corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$, we define $\mathcal{I}(F)(f) = F(f) : F(A_1) \to F(A_2)$, an arrow $F(f) : F(M_1) \to F(M_2)$ in $\mathcal{I}(\mathbf{G}')$. That this assignment is functorial follows from the functoriality of F. ∎

3.15. Corollary. *Construction 3.13 is the object function of a functor*

$$\mathcal{I} : \mathbf{lciGrpd} \to \mathbf{iCat}.$$

Proof. Let $\mathbf{G} \xrightarrow{F} \mathbf{G}' \xrightarrow{G} \mathbf{G}''$ be a composable pair of locally inductive functors. Then, on objects of $\mathcal{I}(\mathbf{G})$ (meet-semilattices forming the partition of $\mathbf{G_0}$),

$$\begin{aligned}\mathcal{I}(G)\mathcal{I}(F)(M) &= \mathcal{I}(G)(M'), \text{ where } M' \text{ such that } FM \subseteq M' \\ &= M'', \text{ where } M'' \text{ such that } M'' \supseteq G(M') = G(FM) = (GF)M \\ &= \mathcal{I}(GF)(M), \text{ by the uniqueness of } M'' \supseteq (GF)M.\end{aligned}$$

Equality of the functors $\mathcal{I}(GF)$ and $\mathcal{I}(G)\mathcal{I}(F)$ follows immediately. That $\mathcal{I}$ preserves identity functors follows from the observation that $\mathcal{I}(1_{\mathbf{G}})(M) = M$ for all objects M in $\mathcal{I}(\mathbf{G})$. ∎

3.16. Theorem. *The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence of categories,*

$$\mathbf{iCat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{lciGrpd}$$

Proof. By Corollary 3.12, the functor $\mathcal{G}$ is fully faithful. We show now that $\mathcal{G}$ is essentially surjective by demonstrating a natural isomorphism $\mathcal{G}\mathcal{I} \cong 1_{\mathbf{lciGrpd}}$.

We start with a locally complete inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i\in I})$ and we consider the composite $\mathcal{G}\mathcal{I}(\mathbf{G})$. Recall that $\mathcal{I}(\mathbf{G})$ has as objects the meet-semilattices M_i and arrows of the form $f : M_1 \to M_2$, where $f : A_1 \to A_2$ is an arrow in $\mathbf{G}$ with $A_1 \in M_1$ and $A_2 \in M_2$. Further recall that every arrow in $\mathbf{G}$ is found exactly once in $\mathcal{I}(\mathbf{G})$. Note that for each object M_i,

$$E_{M_i} = \{\overline{f} : M_i \to M_i | f : M_i \to M_i\} = \{1_{A_i} | A_i \in M_i\} \cong M_i.$$

Then the locally inductive groupoid $\mathcal{G}\mathcal{I}(\mathbf{G})$ contains the following data:

- Objects: $\displaystyle\coprod_{i\in I} E_{M_i} \cong \coprod_{i\in I} M_i = \mathbf{G}_0$.
- Arrows: For each $f : M_1 \to M_2$ in $\mathcal{I}(\mathbf{G})$ corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$, there is an arrow $f : \overline{f} \to \overline{f^\circ} = f : 1_{A_1} \to 1_{A_2} \cong f : A_1 \to A_2$ in $\mathcal{G}\mathcal{I}(\mathbf{G})$. Since arrows of $\mathbf{G}$ are appearing exactly once in $\mathcal{I}(\mathbf{G})$, we have, then, that $(\mathcal{G}\mathcal{I}(\mathbf{G}))_1 \cong \mathbf{G}_1$.
  - Composition: Given two composable arrows corresponding to $f : A_1 \to A_2$ and $g : A_2 \to A_3$ in $\mathcal{G}\mathcal{I}(\mathbf{G})$, we have in $\mathcal{I}(\mathbf{G})$ that

    $$g \circ f = g \star f = [g \,|_* \, A_2 \wedge A_2] \bullet [A_2 \wedge A_2 \, {}_*| \, f] = g \bullet f.$$

    Their composite, then, is

    $$g \star f \text{ in } \mathcal{G}\mathcal{I}(\mathbf{G}) = g \circ f \text{ in } \mathcal{I}(\mathbf{G}) = g \bullet f \text{ in } \mathbf{G}.$$

    That is, composition in $\mathcal{G}\mathcal{I}(\mathbf{G})$ is the same as that in $\mathbf{G}$ up to isomorphism.

– Restrictions: Given an arrow $f : 1_{A_1} \to 1_{A_2} \cong f : A_1 \to A_2$ and $A_1' \leq A_1$, we have that

$$\begin{aligned}(f\,|_*\,A_1') \text{ in } \mathcal{GI}(\mathbf{G}) &\cong f \circ 1_{A_1'} \text{ in } \mathcal{I}(\mathbf{G}) = f \star 1_{A_1'} \text{ in } \mathbf{G} \\ &= [f\,|_*\,A_1 \wedge A_1'] \bullet [A_1 \wedge A_1'\,|_*\,1_{A_1'}] \\ &= [f\,|_*\,A_1'] \bullet 1_{A_1'} = [f\,|_*\,A_1'].\end{aligned}$$

That is, the restrictions of the two ordered groupoids $\mathbf{G}$ and $\mathcal{GI}\mathbf{G}$ are the same up to isomorphism.

This description of $\mathcal{GI}(\mathbf{G})$ is written so that the isomorphism $\mathbf{G} \cong \mathcal{GI}(\mathbf{G})$ follows immediately. ∎

3.17. Note. *In an inverse semigroup $(S, \bullet)$, every idempotent is of the form $s^\bullet \bullet s$ for some $s \in S$. In addition, all idempotents commute. We can then consider the groupoid associated to an inverse semigroup as the Karoubi envelope of the single-object inverse category (with unit) associated to S. In a general inverse category, this fact ensures that every restriction idempotent will appear as an object in the associated locally complete inductive groupoid, and that every object in this groupoid is a restriction idempotent.*

The definition of the functor $\mathcal{G}$ relies on the completeness property of a locally inductive groupoid $\mathcal{G}$ only when defining identities on the meet-semilattices partitioning $\mathbf{G}_0$. Similarly, the identities of an inverse category $\mathbf{X}$ are essential only as top elements of the meet-semilattices E_A. In other words, removing identities from an inverse category is equivalent to removing top elements from the meet-semilattices partitioning a locally inductive groupoid. As a result, the equivalence established in Theorem 3.16 generalizes immediately.

3.18. Corollary. *The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence*

$$\mathbf{isCat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{liGrpd},$$

where **isCat** *is the category of inverse semicategories.*

Since single-object inverse categories are precisely inverse semigroups with identity, it is clear that single-object inverse semicategories are precisely inverse semigroups. With inverse semicategories as multi-object inverse semigroups, we see that Theorem 2.13 – the equivalence between inductive groupoids and inverse semigroups – then follows immediately from Corollary 3.18.

We will end this section with a short discussion on a generalization of Theorem 3.16.

Recall that *prehomomorphisms* of inverse semigroups are functions between inverse semigroups satisfying $\phi(ab) \leq \phi(a)\phi(b)$. Theorem 2.13 can then be generalized to

3.19. Theorem. [Theorem 8, [9]] *The category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors.*

Since the arrows of an inverse category are playing the part of "elements" in each of the "local inverse semigroups", a clear candidate for an inverse categorical analogue arises.

3.20. Definition. *An* oplax functor $F : \mathbf{X} \to \mathbf{X}'$ *of inverse categories consists of the following data:*

- *for each object $A \in \mathbf{X}$, an object $F(A) \in \mathbf{X}'$;*
- *for each arrow $f : A \to B$, an arrow $F(f) : F(A) \to F(B)$ such that*
  - *for each composable pair $f : A \to B$ and $g : B \to C$ in $\mathbf{X}$, $F(gf) \leq F(g)F(f)$, and*
  - *for each object $A \in \mathbf{X}$, $F(1_A) \leq 1_{F(A)}$.*

Clearly, since composition in $\mathcal{G}(\mathbf{X})$ is defined by composition in $\mathbf{X}$, any oplax functor $F : \mathbf{X} \to \mathbf{X}'$ between inverse categories induces an ordered functor $\mathcal{G}(F) : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{X}')$.

Suppose now that $F : \mathbf{G} \to \mathbf{G}'$ is an ordered functor between locally complete inductive groupoids. Recall that composition in $\mathcal{I}(\mathbf{G})$ is defined by the star product in $\mathbf{G}$. Then

$$\begin{aligned} F(g \star f) &= F(g\,|_*\,\mathrm{dom}(g) \wedge \mathrm{cod}(f))F(\mathrm{dom}(g) \wedge \mathrm{cod}(f)\,{}_*|\,f) \\ &= (Fg\,|_*\,F(\mathrm{dom}(g) \wedge \mathrm{cod}(f)))(F(\mathrm{dom}(g) \wedge \mathrm{cod}(f))\,{}_*|\,Ff) \\ &\leq (Fg\,|_*\,F(\mathrm{dom}(g) \wedge F\mathrm{cod}(f))(F\mathrm{dom}(g) \wedge F\mathrm{cod}(f)\,{}_*|\,Ff) \\ &= Fg \star Ff \end{aligned}$$

and thus F induces an oplax functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{G}')$. Specifically, since the identities in $\mathcal{I}(\mathbf{G})$ are the top elements of $\mathbf{G}$, $\mathcal{I}(F)$ is strict on identities.

These arguments can then be easily extended to prove the following.

3.21. Theorem. *The category of locally complete inductive groupoids and ordered functors is equivalent to the category of inverse categories and oplax functors.*

3.22. Note. *Since the 2-category structure of an inverse category is posetal, pseudofunctors (oplax functors whose 2-cells are isomorphisms) are exactly ordinary functors between inverse categories. The category of inverse categories and pseudofunctors is therefore equal to the category of inverse categories and ordinary functors, and equivalent to the category of locally complete inductive groupoids and locally inductive functors.*

References

[1] R. Cockett and S. Lack. Restriction categories I: categories of partial maps. *Theoretical Computer Science*, 270:223–259, 2002.

[2] C. Ehresmann. Gattungen von lokalen Strukturen. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 60:49–77, 1958.

[3] C. Ehresmann. Catégories inductives et pseudogroupes. *Annales de lInstitut Fourier*, 10:307–332, 1960.

[4] V. Gould and C. Hollings. Restriction semigroups and inductive constellations. *Communications in Algebra*, 38:261–287, 2010.

[5] Christopher Hollings. The Ehresmann–Schein–Nambooripad Theorem and its successors. *European Journal of Pure and Applied Mathematics*, pages 1–39, 2009.

[6] Christopher Hollings. Extending the Ehresmann-Schein-Nambooripad theorem. *Semigroup Forum*, 80(3):453–476, 2010.

[7] J. Kastl. *Inverse Categories*, volume 7 of *Studien zur Algebra und ihre Anwendungen*. Akademie-Verlag, 1979.

[8] M. V. Lawson. Semigroups and ordered categories. I. The reduced case. *J. Algebra*, 141(2):422–462, 1991.

[9] M.V. Lawson. *Inverse Semigroups: The Theory of Partial Symmetries*. World Scientific Publishing Co., 1998.

[10] Markus Linckelmann. On inverse categories and transfer in cohomology. *Proceedings of the Edinburgh Mathematical Society. Series II*, 56(1):187–210, 2013.

[11] K.S.S. Nambooripad. *Structure of Regular Semigroups*. PhD thesis, University of Kerala, 1973.

[12] K.S.S. Nambooripad. Structure of regular semigroups, I. Fundamental regular semigroups. *Semigroup Forum*, 9(1):354–363, 1975.

[13] K.S.S. Nambooripad. Structure of regular semigroups, II. The general case. *Semigroup Forum*, 9(1):364–371, 1975.

[14] K.S.S. Nambooripad. Structure of regular semigroups. In *Memoirs of the American Mathematical Society*. American Mathematical Society, 1979.

[15] G.B. Preston. *Some Problems in the Theory of Ideals*. PhD thesis, University of Oxford, 1953.

[16] Pedro Resende. *Lectures on étale groupoids, inverse semigroups and quantales*. 2006.

[17] B.M. Schein. On the theory of inverse semigroups and generalised groups. *American Mathematical Society Translations*, 2(113):89–122, 1979.

Department of Mathematics, Statistics and Computer Science
St. Francis Xavier University
2323 Notre Dame Lane
Antigonish, NS B2G 1N5
Canada

Department of Mathematics and Statistics
Dalhousie University
Halifax, NS B3H 4R2
Canada

Email: ddewolf@stfx.ca
dorette.pronk@dal.ca