The Ehresmann-Schein-Nampooripad Theorem for Inverse Categories

Darien DeWolf [1] and Dorette Pronk [2]

Abstract

The Ehresmann-Schein-Nampooripad (ESN) Theorem asserts an equivalence between the category of inverse semigroups and the category of inductive groupoids. In this paper, we consider the category of inverse categories and functors – a natural generalization of inverse semigroups – and extend the ESN theorem to an equivalence between this category and the category of top-heavy locally inductive groupoids and locally inductive functors. From the proof of this extension, we also generalize the ESN Theorem to an equivalence between the category of inverse semicategories and the category of locally inductive groupoids and to an equivalence between the category of inverse categories with oplax functors and the category of top-heavy locally inductive groupoids and ordered functors.

Keywords: Inverse Semigroup, Inverse Category, Inductive Groupoid, Top-Heavy Locally Inductive Groupoid, Inverse Semicategory

1 Introduction

The Ehresmann-Schein-Nampooripad (ESN) Theorem asserts the existence of an equivalence between the category of inverse semigroups (with semigroup homomorphisms) and the category of inductive groupoids (with inductive functors). A groupoid is called ordered in this context if there is a compatible (functorial) order on both objects and arrows with a notion of restriction on the arrows such that an arrow $f\colon A \to B$ has a unique restriction $f'\colon A' \to B'$ with $f' \leq f$ for any object $A' \leq A$. For the precise definition see, Definition 1.1. Ordered functors between these are functors that preserve the order. Furthermore, an ordered groupoid is called inductive when the objects form a meet-semilattice and an ordered functor is inductive when it preserves the meets. The correspondence of the ESN Theorem is directly extendable to inverse semigroups and prehomomorphisms when one takes ordered functors, rather than inductive functors, between the inductive groupoids.

This theorem has been extended to various larger classes of semigroups such as regular semigroups [7, 8, 9], two-sided restriction semigroups (also called Ehresmann semigroups) [4] and more general restriction groups [3] with either semigroup homomorphisms or $\wedge$- or $\vee$-prehomomorphisms. The main ideas in this context have focused on either changing the requirement for a meet-semilattice structure to a different order structure on the objects of the groupoid, or on generalizing to inductive categories rather than groupoids.

Our approach here is to generalize this equivalence in a different direction. Semigroups can be viewed as single-object semicategories and we want to obtain a 'multi-object' version of the correspondence. As groupoids can be thought of as the multi-object version of groups, we think of inverse categories as a multi-object version of inverse semigroups. In this paper, we prove a new generalization of the ESN theorem which extends the result to inverse categories. Since we are generalizing the concept of *inverse* semigroup, we will remain within the category of groupoids. They will still be ordered, but the order structure will only be locally inductive in a suitable sense: the objects need to form a disjoint union of meet-semilattices. Since inverse categories have units, we further require that the meet-semilattices have a top-element. If we instead generalize to inverse semicategories, this requirement is not needed. Locally inductive functors, ordered functors that preserve all meets that exist, will correspond to functors of

[1]Email: ddewolf@dal.ca
[2]Email: dorette.pronk@dal.ca
Mathematics Department, Dalhousie University, Halifax, Nova Scotia , Canada, B3H
Both authors thank NSERC for its funding of this research.

inverse semicategories (Corollary 2.16). We will also show that the category of inverse categories and oplax functors is equivalent to the category of top-heavy locally inductive groupoids and locally inductive functors, generalizing the classical result that the category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors (Theorem 2.19).

The groupoid we construct for an inverse category was independently considered in the work of Linckelmann [6] on category algebras. Linckelmann observes that this groupoid has the same category algebra as the original inverse category, giving the category algebra of an inverse category the structure of a groupoid algebra: a groupoid algebra over a commutative ring is a direct product of matrix rings. In this paper, we introduce this groupoid with an ordered structure and observe the important characterizing properties of the order structure to obtain an equivalence of categories between the category of inverse categories and the category of these top-heavy locally inductive groupoids.

From the semigroup perspective, this begs the question of whether there are appropriate multi-object versions of the other classes of semigroups mentioned above which then may be shown to be equivalent to appropriate versions of locally inductive categories.

1.1 Inductive Groupoids and Inverse Semigroups

Inductive groupoids are a class of groupoids whose arrows are equipped with a partial order satisfying certain conditions and whose objects form a meet-semilattice. Charles Ehresmann used ordered groupoids to model pseudogroups while inverse semigroups, introduced by Gordon Preston [10], were concurrently used as an alternate model for pseudogroups. Ehresmann was certainly aware of the connection between ordered (inductive) groupoids and inverse semigroups, as it was Ehresmann who first introduced the tensor product required to make the correspondence work. Boris Schein [11] made this connection explicit, requiring that the set of objects form a meet-semilattice, thus guaranteeing the existence of this tensor product for all arrows of the groupoid. K.S.S. Nambooripad [7, 8, 9] independently developed the theory of so-called regular systems and their correspondence to so-called regular groupoids. This theory is, in fact, more general and specializes to the correspondence of inverse semigroups to inductive groupoids. A more detailed history of inverse semigroups, inductive groupoids and their applications can be found in Hollings' [2]. In this section, we present the modern exposition of this correspondence, which can be found in Mark Lawson's book [5], where these constructions and their equivalence were first explicitly given.

Definition 1.1. A groupoid $(G, \circ)$ is said to be an *ordered groupoid* whenever there is a partial order $\leq$ on its arrows satisfying the following four conditions:

- (i) For each arrow $f, g \in G$, $f \leq g$ implies $f^{-1} \leq g^{-1}$.
- (ii) For each arrow $a, A, b, B \in G$ such that $a \leq A$, $b \leq B$ and the composites ab and AB exist, then $ab \leq AB$.
- (iii) For each arrow $f : A' \to B$ in G and objects $A \leq A'$ in G, there exists a unique *restriction of f to A*, denoted $[f|_* A]$, such that $\mathrm{dom}[f|_* A] = A$ and $[f|_* A] \leq f$.
- (iv) For each arrow $f : A \to B'$ in G and objects $B \leq B'$ in G, there exists a unique *corestriction of f to B*, denoted $[B\,_*|f]$, such that $\mathrm{cod}[B\,_*|f] = B$ and $[B\,_*|f] \leq f$.

An ordered groupoid is said to be an *inductive groupoid* whenever its objects form a meet-semilattice.

Though it is sometimes convenient to explicitly give both the restrictions and corestrictions in an ordered groupoid, the following proposition makes it necessary only to include one of them in any proofs.

Proposition 1.2 ([5])**.** *In Definition 1.1, conditions (iii) and (iv) are equivalent.*

Definition 1.3. Let $\mathbf{G}$ be an ordered groupoid with arrows $\alpha, \beta \in \mathbf{G}$. If $\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)$ exists, the *tensor product* $\alpha \otimes \beta$ of α and β is defined as

$$\alpha \otimes \beta = [\alpha \,|_*\, \mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)][\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)\,_*|\, \beta].$$

Proposition 1.4 ([5])**.** *This tensor product is associative whenever it exists. In addition, this tensor product admits pseudoinverses given by the inverses in the ordered groupoid, making $(\mathbf{G}_1, \otimes)$ an inverse semigroup.* □

Proposition 1.5. *For all objects $A \leq B$ of an ordered groupoid, $[1_B \,|_*\, A] = 1_A = [A \,|_*\, 1_B]$.*

Proof. Since the partial order on arrows induces the partial order on the objects of an ordered groupoid and the objects of a category are identified by the identity arrow on that object, we have that $1_A \leq 1_B$. Since the (co)domain of 1_A is A, we have $[1_B \,|_*\, A] = 1_A = [A \,|_*\, 1_B]$ by the uniqueness of (co)restrictions □

Definition 1.6. A morphism $F : \mathbf{G} \to \mathbf{H}$ of ordered groupoids (an *ordered functor*) is a functor such that, for all arrows $f \leq g$ in $\mathbf{G}$, $F(f) \leq F(g)$ in $\mathbf{H}$. An ordered functor between inductive groupoids is said to be *inductive* whenever it preserves the meet structure on objects.

Notation. We denote the category of ordered groupoids and ordered functors by **OGrpd** and the category of inductive groupoids and inductive functors by **IGrpd**.

We will now briefly review Lawson's description of functorial constructions that form the equivalence of categories between the category of inverse semigroups and the category of inductive groupoids. We remind the reader that full details can be found in [5].

Construction 1.7 (Inverse Semigroups to Inductive Groupoids)**.** Given an inverse semigroup $(S, \bullet)$, define an inductive groupoid $\mathcal{G}(S)$ with the following data:

- Objects: $\mathcal{G}(S)_0 = E(S)$, the idempotents in S. Since S is an inverse semigroup, $E(S)$ is a meet-semilattice with meets given by the product in S.
- Arrows: For each element $s \in S$, there is an arrow $s : s^\bullet s \to ss^\bullet$. Composition is given by multiplication in S and identities are the elements of $E(S)$.
- Inverses: For each arrow $s : s^\bullet s \to ss^\bullet$ in $\mathcal{G}(S)$, define $s^{-1} = s^\bullet$, its pseudoinverse in S.
- The partial order on arrows is given by the natural partial order ($s \leq t$ if and only if $s = te$ for some idempotent e) on the elements of S. It can be checked that this partial order satisfies conditions (i) and (ii) of an ordered groupoid.
- The (co)restrictions are also given by multiplication in S. This can be checked to satisfy condition (iii) of an ordered groupoid.

Construction 1.8 (Inductive Groupoids to Inverse Semigroups)**.** Given an inductive groupoid $(G, \circ, \leq)$, define an inverse semigroup $\mathcal{S}(G)$ whose elements are the arrows of G and whose multiplication is given by the tensor product. This is an inverse semigroup operation with inverses those from G (Proposition 1.4).

Theorem 1.9 (ESN, [5])**.** *The constructions $\mathcal{G}$ and $\mathcal{S}$ are functorial and form an equivalence of categories*

$$\mathbf{IGrpd} \underset{\mathcal{G}}{\overset{\mathcal{S}}{\rightleftarrows}} \mathbf{IS}$$

□

1.2 Inverse Categories

As groupoids are for groups, we seek a structure describing multi-object inverse semigroups. Inverse semigroups with units are exactly single-object inverse categories, so it seems that inverse (semi)categories could be appropriate for such a role.

Definition 1.10. A category $\mathbf{X}$ is said to be an *inverse category* whenever, for each arrow $f : A \to B$ in $\mathbf{X}$, there exists a unique $f^\circ : B \to A$ in $\mathbf{X}$ such that $f \circ f^\circ \circ f = f$ and $f^\circ \circ f \circ f^\circ = f^\circ$.

Motivated, perhaps, by Ehresmann's use of inductive groupoids to model pseudogroups, we may gain some intuition about (finite) inverse semigroups by thinking of partial automorphisms on a set. Cockett and Lack [1] introduced the notion of a *restriction category* to give an axiomatic treatment to partiality of maps in a category.

Definition 1.11. A restriction structure on a category $\mathbf{X}$ is an assignment of an arrow $\overline{f_A} : A \to A$ to each arrow $f : A \to B$ in $\mathbf{X}$ satisfying the following four conditions:

(R.1) For all maps f, $f\,\overline{f_A} = f$.

(R.2) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f_A}\,\overline{g_A} = \overline{g_A}\,\overline{f_A}$.

(R.3) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g_A\,\overline{f_A}} = \overline{g_A}\,\overline{f_A}$.

(R.4) For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g_A}\,f = f\,\overline{(gf)_B}$.

A category equipped with a restriction structure is called a *restriction category*.

Restriction categories reduce deduction about the partiality of a map to algebraic manipulation. Intuitively, we may think of $\overline{f_A}$ as the "domain of definedness" (or, the subobject of A on which f is defined) for a morphism f in $\mathbf{X}$.

Notation. We may write $\overline{f}$ instead of $\overline{f_A}$ if the explicit statement of domain adds no useful information.

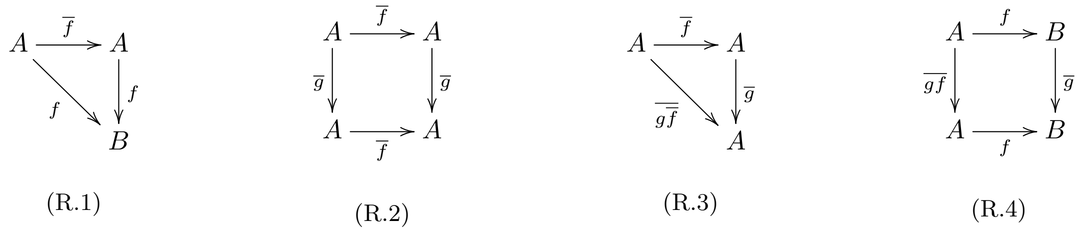

Figure 1: The axioms of a restriction category, diagrammatically.

Example 1.12. Examples of restriction categories:

(a) This is the prototypical example of a restriction category. Let Par denote the category of sets and partial functions. Par is a restriction category by defining, for each partial function $f : A \to B$, and each $x \in A$,

$$\overline{f}(x) = \begin{cases} x & f(x) \text{ is defined} \\ \text{not defined} & \text{otherwise} \end{cases}$$

The axioms (R.1) – (R.4) required for Par to be a restriction category are easily verified. We can think of $\overline{f}$ as the "domain of definedness" of f and can interpret expressions such as $f\overline{g}$ as "f restricted to where g is defined".

(b) Let **C** be an ordinary category equipped with a stable sytem $\mathcal{M}$ of monics (all details of this example can be found in [1]). Define a category Par($\mathbf{C}, \mathcal{M}$) with the following data:

- Objects: Same objects as **C**.
- Arrows: Isomorphism classes of spans

$$X \xleftarrow{i} D \xrightarrow{f} Y,$$

  where $i \in \mathcal{M}$. We will sometimes denote such an arrow (actually, its isomorphism class) as (i, f).
- Composition: Composition is given by pullback.
- Restrictions: Given any arrow (i, f), the assignment $\overline{(i, f)} = (i, i)$ defines a restriction structure on Par($\mathbf{C}, \mathcal{M}$).

The next lemma lists some useful identities that will be used, without reference, to make calculations about restriction categories.

Lemma 1.13 ([1])**.** *If* **X** *is a restriction category, then:*

(i) $\overline{f}$ *is idempotent;*

(ii) $\overline{f}\,\overline{gf} = \overline{gf}$;

(iii) $\overline{\overline{g}f} = \overline{gf}$;

(iv) $\overline{\overline{f}} = \overline{f}$;

(v) $\overline{g\overline{f}} = \overline{g}\,\overline{f}$;

(vi) if f *is monic, then* $\overline{f} = 1$;

(vii) $f\overline{g} = f$ *implies* $\overline{f} = \overline{f}\overline{g}$. □

Note. A restriction category **X** has a natural, locally partially ordered 2-category structure: for any two parallel arrows $f, g : C \to D$ in **X**, we define a partial order by $f \leq g$ if and only if $f = g\overline{f}$. Notice that if $f \leq g$, then

$$\overline{g}\,\overline{\overline{f}} = \overline{g}\,\overline{f} = \overline{g\overline{f}} = \overline{f}$$

and thus $\overline{f} \leq \overline{g}$.

Proposition 1.14. *Suppose that a, A, b and B are arrows in a restriction category $\mathbf{X}$ with $a \leq A$ and $b \leq B$. If the composites ab and AB exist, then $ab \leq AB$.*

Proof. Suppose that a, A, b and B are arrows in $\mathbf{X}$ with $a \leq A$, $b \leq B$ and such that the composites ab and AB exist. Then

$$AB\overline{ab} = AB\overline{b}\,\overline{ab} = Ab\overline{ab} = A\overline{a}b = ab$$

and thus $ab \leq AB$. □

Definition 1.15. A map f in a restriction category $\mathbf{X}$ is called *total* whenever $\overline{f} = 1$.

Lemma 1.16 ([1])**.** *If $\mathbf{X}$ is a restriction category, then:*

(i) every monomorphism is total;

(ii) if f and g are total, then gf is total;

(iii) if gf is total, then f is total;

(iv) the total maps form a subcategory, denoted $\mathrm{Tot}(\mathbf{X})$. □

Definition 1.17. A morphism $F : \mathbf{X} \to \mathbf{Y}$ of restriction categories (a *restriction functor*) is a functor such that $F\left(\overline{f}\right) = \overline{F(f)}$ for each $f \in X_1$.

Definition 1.18. A map f in a restriction category $\mathbf{X}$ is called a *restricted isomorphism* whenever there exists a map g – called a *restricted inverse* of f – such that $gf = \overline{f}$ and $fg = \overline{g}$.

Following from the commutation of idempotents (Restriction Category Axiom 1.11), we have the following property of restricted isomorphisms:

Theorem 1.19 (Lemma 2.18(vii), [1])**.** *If f is a restricted isomorphism, then its restricted inverse is necessarily unique.*

Notation. Given a map f in a restriction category $\mathbf{X}$, we denote its restricted inverse (if it exists) by f°.

Definition 1.20. A restriction category $\mathbf{X}$ is called an *inverse category*, whenever every map f is a restricted isomorphism.

Example 1.21. Some inverse categories:

(a) The category of sets and partial bijections.

(b) Any inverse semigroup with unit is a single-object inverse category.

(c) Any groupoid is an inverse category with all arrows total.

Lemma 1.22 ([1])**.** *If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, then F preserves*

(i) total maps,

(ii) restriction idempotents,

(iii) restricted sections and

(iv) restricted isomorphisms.

Note. Any functor between inverse categories is a restriction functor preserving restricted isomorphisms. This follows from the restriction structure and restricted isomorphisms being defined as specific composites. We will therefore omit the words "inverse" and "restriction" when speaking of functors between inverse categories.

As expected, restriction idempotents are their own restricted inverse.

Proposition 1.23. *In an inverse category, $\left(\overline{f}\right)^\circ = \overline{f}$ for all arrows f.*

Proof. Since all arrows in an inverse category are restricted isomorphisms,

$$\left(\overline{f}\right)^\circ = (f^\circ f)^\circ = f^\circ (f^\circ)^\circ = f^\circ f = \overline{f}.$$ □

It is clear that inverse categories, interpreted as restriction categories in Definition 1.20, are exactly the same as inverse categories interpreted as multi-object inverse semigroups in Definition 1.10; that is restrictions come for free in an inverse category and are given by $\overline{f} = f^\circ f$. In this paper, we choose to think in terms of restriction categories for two reasons: firstly, the choice of notation in restriction categories facilitates calculations. Secondly, we prefer to think of (finite) inverse semigroups as collections of partial automorphisms on a (finite) set whose idempotents are partial identities – inverse categories in terms of restriction categories explicitly make use of this intuition.

Notation. We denote the category of inverse categories and functors by **ICat**.

2 Main Result

In this section, we introduce the notion of top-heavy locally inductive groupoids: ordered groupoids whose objects may be partitioned into meet-semilattices, each of which contain a top element. We will then give functorial constructions of top-heavy locally inductive groupoids from inverse categories, and vice versa. These constructions will then be seen to give an equivalence of categories between **ICat** and **TLIGrpd**. The identities of an inverse category are seen to correspond to the tops of the meet-semilattices in a top-heavy locally inductive groupoid and the equivalence can thus be immediately generalized to an equivalence between the category of inverse semicategories and semifunctors and the category of locally inductive groupoids and locally inductive functors. Finally, we end this paper with a short discussion of a categorical analogue of the classical result in semigroup theory that the category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors. Explicitly, we show that the category of inverse categories and oplax functors is equivalent to the category of top-heavy locally inductive groupoids and ordered functors.

Definition 2.1. Let A be an object of a restriction category $\mathbf{X}$. Let E_A denote the set of restrictions of all endomorphisms on A. That is,

$$E_A = \left\{ \overline{f_A} : A \to A | f : A \to A \in \mathbf{X} \right\}.$$

Notice that, for any $f : A \to B$ in $\mathbf{X}$, we have $\overline{f} : A \to A \in E_A$, since $\overline{\overline{f}} = \overline{f}$. The reason for specifying that the restrictions in E_A come from endomorphisms in $\mathbf{X}$, then serves no use further than simply reminding us that the equivalence we are trying to establish here is based on the observation that an inverse category is, at each object, an inverse semigroup (with unit).

Proposition 2.2. *For each object A of a restriction category $\mathbf{X}$, E_A is a meet-semilattice with meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$. In addition, E_A has top element 1_A.*

Proof. First of all, E_A is a poset with the natural partial order inherited from $\mathbf{X}$. We now show that E_A has finite meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$:

- First, it is a lower bound:
$$\overline{a}\,\overline{\overline{a} \wedge \overline{b}} = \overline{a}\,\overline{\overline{a}\overline{b}} = \overline{a}\,\overline{a}\overline{b} = \overline{a}\,\overline{b} = \overline{a} \wedge \overline{b}$$
and thus $\overline{a} \wedge \overline{b} \leq \overline{a}$. Similarly, $\overline{a} \wedge \overline{b} \leq \overline{b}$.
- This lower bound is unique up to isomorphism (equality): suppose that $\overline{d}$ is such that $\overline{d} \leq \overline{a}$, $\overline{d} \leq \overline{b}$ and $\overline{a} \wedge \overline{b} \leq \overline{d}$. Then
$$\overline{d} = \overline{a}\,\overline{d} = \overline{a}\,\overline{b}\,\overline{d} = \overline{d}\,\overline{\overline{a}\,\overline{b}} = \overline{d}\,\overline{\overline{a} \wedge \overline{b}} = \overline{a} \wedge \overline{b}.$$

Finally, since $\overline{1_A} = 1_A$, $1_A \in E_A$. Also, given any $\overline{f} : A \to A$, $1_A\overline{f} = \overline{f}$ and thus $\overline{f} \leq 1_A$ and 1_A is the top element of E_A. □

Proposition 2.3. *For each pair of objects A and B of a restriction category $\mathbf{X}$, if $A \neq B$, then*

$$E_A \cap E_B = \varnothing.$$

Proof. If $\overline{f} \in E_A \cap E_B$, then $A = \mathrm{dom}\left(\overline{f}\right) = B$. □

We may now give the (functorial) constructions giving an equivalence between the category of top-heavy locally inductive groupoids and inverse categories.

Construction 2.4. Given an inverse category $\left(\mathbf{X}, \circ, \overline{(-)}\right)$, define a groupoid $(\mathcal{G}(\mathbf{X}), \bullet, \leq)$ with the following data:

- Objects: $\mathcal{G}(\mathbf{X})_0 = \coprod_{A \in \mathbf{X_0}} E_A$.
- Arrows: Every arrow in $\mathcal{G}(\mathbf{X})$ is of the form $f : \overline{f_A} \to \overline{f_B^\circ}$ for each arrow $f : A \to B$ in $\mathbf{X}$.
  - Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$. This composite is indeed an arrow, for
  $$\overline{gf} = \overline{\overline{g}f} = \overline{\overline{f^\circ}f} = \overline{f}$$
  and
  $$\overline{(gf)^\circ} = \overline{f^\circ g^\circ} = \overline{\overline{f^\circ}g^\circ} = \overline{\overline{g}g^\circ} = \overline{g^\circ}.$$

– Identities: For any object $\overline{f} : A \to A$ in $\mathcal{G}(\mathbf{X})$, define $1_{\overline{f}} = \overline{f}$ (which is well-defined since $\overline{\overline{f}} = \overline{f}$). The identity then satisfies the appropriate axiom: for each $g : \overline{g} \to \overline{g^\circ}$ with $\overline{g} = \overline{f}$ and $\overline{g^\circ} = \overline{f^\circ}$, we have $\overline{f^\circ} g = \overline{g^\circ} g = g$ and $g\overline{f} = g\overline{g} = g$.
– Inverses: Given an arrow $f : \overline{f} \to \overline{f^\circ}$, define $f^{-1} : \overline{f^\circ} \to \overline{f}$ to be f°, the unique restricted inverse of f from $\mathbf{X}$'s inverse structure. The composites are $ff^\circ = \overline{f^\circ} = 1_{\overline{f^\circ}}$ and $f^\circ f = \overline{f} = 1_{\overline{f}}$ as required.

Definition 2.5. An ordered groupoid is said to be a *locally inductive groupoid* whenever there is a partition $\{M_i\}_{i \in I}$ of $\mathbf{G}_0$ into meet-semilattices M_i with the property that any two comparable objects be in the same meet-semilattice M_i. A locally inductive groupoid is said to be *top-heavy* whenever each meet-semilattice M_i admits a top-element $\top_i$.

Note. The requirement that any two comparable objects of a locally inductive groupoid be in the same meet-semilattice is corresponding to our intuition that if the meet $A \wedge B$ of two objects A and B exists in M_i, then A and B, both sitting above this meet, should also be elements of M_i.

Definition 2.6. An ordered functor between locally inductive groupoids is said to be *locally inductive* whenever it preserves all meets that exist. In particular, a locally inductive functor will preserve empty meets and thus top elements and there is no requirement to define so-called "top-heavy locally inductive functors".

Notation. We denote the category of locally inductive groupoids and locally inductive functors by **LIGrpd** and the category of top-heavy locally inductive groupoids and locally inductive functors by **TLIGrpd**.

Proposition 2.7. *For all inverse categories* $\mathbf{X}$, $\mathcal{G}(\mathbf{X})$ *is a top-heavy locally inductive groupoid.*

Proof. Recall that the partial order on the objects $\overline{f}$ in $\mathcal{G}(\mathbf{X})$ is that which is induced by the partial order on the arrows of $\mathbf{X}$. That is, $\overline{f} \leq \overline{g}$ if and only if $\overline{f} = \overline{g\overline{f}} = \overline{g}\,\overline{f}$. We now prove that this partial order gives $\mathcal{G}(\mathbf{X})$ the structure of an ordered groupoid:

(i) Suppose that f and g are arrows in $\mathcal{G}(\mathbf{X})$ with $f \leq g$. That is, we suppose that $g\overline{f} = f$ (since these are also arrows in $\mathbf{X}$). Then

$$\begin{aligned} f^\circ &= (g\overline{f})^\circ = \overline{f}^\circ g^\circ = \overline{f} g^\circ = \overline{f} g^\circ g g^\circ = g^\circ g \overline{f} g^\circ \\ &= g^\circ g \overline{f}\,\overline{f} g^\circ = g^\circ g \overline{f}\,\overline{f}^\circ g^\circ = g^\circ g \overline{f} (g\overline{f})^\circ = g^\circ f f^\circ \\ &= g^\circ \overline{f^\circ} \end{aligned}$$

and thus $f^{-1} = f^\circ \leq g^\circ = g^{-1}$.

(ii) This follows directly from Proposition 1.14.

(iii) Given an arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ with an object $\overline{e} \leq \overline{\alpha}$, we define the restriction $[\alpha|_*\overline{e}]$ of α to $\overline{e}$ to be $\alpha\overline{e}$. This is indeed an arrow whose domain is $\overline{e} : \overline{\alpha\overline{e}} = \overline{\alpha}\,\overline{e} = \overline{e}$.
Also, $\alpha\overline{\alpha\overline{e}} = \alpha\overline{\alpha}\,\overline{e} = \alpha\overline{e}$, so that $\alpha\overline{e} \leq \alpha$.
If $\beta \leq \alpha$ is any other arrow with $\mathrm{dom}(\beta) = \overline{e}$, we have $\alpha\overline{\beta} = \beta$ and $\overline{\beta} = \overline{e}$, so that $\beta = \alpha\overline{e}$ and thus $[\alpha|_*\overline{e}]$ as defined is unique.

(iv) Given an arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ with an object $\overline{e} \leq \overline{\alpha^\circ}$, we define the corestriction $[\overline{e}\,{}_*|\alpha]$ of α to $\overline{e}$ to be $\overline{e}\alpha$. This is indeed an arrow whose codomain is $\overline{e} : \overline{(\overline{e}\alpha)^\circ} = \overline{\alpha^\circ\overline{e}} = \overline{a^\circ}\,\overline{e} = \overline{e}$.
Also, $\alpha\overline{\overline{e}\alpha} = \alpha\overline{e\alpha} = \alpha(e\alpha)^\circ e\alpha = \alpha\alpha^\circ e^\circ e\alpha = e^\circ e\alpha\alpha^\circ\alpha = \overline{e}\alpha$, so that $\overline{e}\alpha \leq \alpha$.
If $\beta \leq \alpha$ is any other arrow with $\mathrm{cod}(\beta) = \overline{e}$, we have $\beta^\circ \leq \alpha^\circ$ (property (i) of ordered groupoids) and thus $\alpha^\circ\overline{\beta^\circ} = \beta^\circ$ and $\overline{\beta^\circ} = \overline{e}$, so that $\beta^\circ = \alpha^\circ\overline{e} = (\overline{e}\alpha)^\circ$ and thus $[\overline{e}\,{}_*|\alpha]$ as defined is unique.

Given the choice of objects for $\mathcal{G}(\mathbf{X})$, it follows immediately from Propositions 2.2 and 2.3 that $\mathcal{G}(\mathbf{X})$ is a top-heavy locally inductive groupoid. □

The composition in $\mathcal{G}(\mathbf{X})$ of f and g exists exactly when $\overline{f} = \overline{g^\circ}$ and is defined by the composition in $\mathbf{X}$. The tensor product in $\mathcal{G}(\mathbf{X})$ is a natural extension of this composition in the sense that it exists whenever the meet $\overline{f} \wedge \overline{g^\circ}$ exists. This lemma shows that this extension is also defined by the composition in $\mathbf{X}$.

Lemma 2.8. *If* $\mathbf{X}$ *is an inverse category, then in* $\mathcal{G}(\mathbf{X})$ *the tensor products (when defined) are given by composition in* $\mathbf{X}$:

$$f \otimes g = fg.$$

Proof. Recall that, for any arrow f in $\mathbf{X}$, $\mathrm{dom}(f) = \overline{f}$ and $\mathrm{cod}(f) = \overline{f^\circ}$. Then

$$\begin{aligned} f \otimes g &= [f\,|_* \,\mathrm{dom}(f) \wedge \mathrm{cod}(g)]\,[\mathrm{dom}(f) \wedge \mathrm{cod}(g)\,{}_*|\,g] \\ &= \left[f\,|_*\,\overline{f} \wedge \overline{g^\circ}\right]\left[\overline{f} \wedge \overline{g^\circ}\,{}_*|\,g\right] = \left[f\,|_*\,\overline{f}\,\overline{g^\circ}\right]\left[\overline{f}\,\overline{g^\circ}\,{}_*|\,g\right] = f\overline{f}\,\overline{g^\circ}\,\overline{f}\,\overline{g^\circ}g = f\overline{f}\,\overline{g^\circ}g = fg \end{aligned}$$ □

Proposition 2.9. *Locally inductive functors preserve tensors.*

Proof. This follows immediately from the definition of a locally inductive functor and the fact that any ordered functor preserves restrictions and corestrictions [5, Proposition 4.1.2(1)]. □

Proposition 2.10. *For each functor $F : \mathbf{X} \to \mathbf{Y}$ between inverse categories, there exists a locally inductive functor $\mathcal{G}(F) : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{Y})$.*

Proof. We claim that $F : \mathbf{X} \to \mathbf{Y}$ induces a locally inductive functor $\mathcal{G}(F)$ between the groupoids $\mathcal{G}(\mathbf{X})$ and $\mathcal{G}(\mathbf{Y})$. Since F is a functor of inverse categories, we have, for each $\overline{f}$ in $\mathbf{X}$, that $F\overline{f} = \overline{F(f)}$ is a restriction idempotent in $\mathbf{Y}$. We can then define, for any object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, $\mathcal{G}(F)(\overline{f}) = F\overline{f}$ and this is a well-defined object function.

Given an arrow $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$, we define

$$\mathcal{G}(F)(f) := \left[F(f) : F\left(\overline{f}\right) \to F\left(\overline{f^\circ}\right)\right] = \left[F(f) : \overline{F(f)} \to \overline{F(f^\circ)}\right].$$

We check that this is indeed an arrow in $\mathcal{G}(\mathbf{Y})$. Clearly, $F(f)$ has the correct domain. We check, then, that it has the correct codomain; that is, we verify that $\overline{(F(f))^\circ} = \overline{F(f^\circ)}$. By Lemma 1.22(iv), $(F(f))^\circ = F(f^\circ)$. It follows, then, that $\overline{(F(f))^\circ} = \overline{F(f^\circ)}$ and thus F is well defined on arrows.

Since the objects of $\mathcal{G}(\mathbf{X})$ are specific arrows in $\mathbf{X}$ and the composition in $\mathcal{G}(\mathbf{X})$ is, when defined, given by composition in $\mathbf{X}$, the functoriality of $\mathcal{G}(F)$ follows from the functoriality of F.

We check now that F is an ordered functor. That is, we must check that F preserves partial orders. Suppose that $f \leq g$ are arrows in $\mathcal{G}(\mathbf{X})$. Then $g\overline{f} = \overline{f}$ and thus

$$F(g)\overline{F(f)} = F(g)F(\overline{f}) = F(g\overline{f}) = F(\overline{f}) = \overline{F(f)}.$$

Therefore, $F(f) \leq F(g)$ in $\mathcal{G}(\mathbf{Y})$ and F is an ordered functor.

Finally, we verify that F is a locally inductive functor. If $\overline{a} \wedge \overline{b}$ exists in $\mathcal{G}(\mathbf{X})$, then $\overline{a}$ and $\overline{b}$ are endomorphisms on the same object and are thus composable and in the same meet-semilattice. Then, by the functoriality of F, $F\left(\overline{a} \wedge \overline{b}\right) = F\left(\overline{a}\,\overline{b}\right) = F(\overline{a})F\left(\overline{b}\right) = F(\overline{a}) \wedge F\left(\overline{b}\right)$. □

Corollary 2.11. *Construction 2.4 is the object function of a fully faithful functor $\mathcal{G} : \mathbf{ICat} \to \mathbf{TLIGrpd}$.*

Proof. By the proof of Proposition 2.10, $\mathcal{G}$ is clearly a faithful functor.

Now, let $\mathbf{X}$ and $\mathbf{X}'$ be inverse categories and suppose that $F : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{X}')$ is a locally inductive functor. We seek, then, a functor $F' : \mathbf{X} \to \mathbf{X}'$ with $\mathcal{G}(F') = F$.

For any two restriction idempotents $\overline{e}$ and $\overline{f}$ in E_A, we have $F(\overline{e} \wedge \overline{f}) = F\overline{e} \wedge F\overline{f}$ since F is locally inductive. This implies that $F\overline{e}$ and $F\overline{f}$ are $\mathbf{X}'$-endomorphisms on the same object and thus $F(E_A) \subseteq E_B$ for some object $B \in \mathbf{X}'$. So we can define, for each object $A \in \mathbf{X}$, $F'(A)$ to be the object in $\mathbf{X}'$ satisfying $F(E_A) \subseteq E_{F'(A)}$ in $\mathcal{G}(\mathbf{X}')$.

Given any arrow $f : A \to B$ in $\mathbf{X}$, we must define an arrow $F'(f) : F'(A) \to F'(B)$ in $\mathbf{X}'$. We know that f corresponds to the arrow $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$, whose image under F is $F(f) : F\overline{f} \to F\overline{f^\circ}$ in $\mathcal{G}(\mathbf{X}')$. Since $F\overline{f} \in F(E_A)$ and $F\overline{f^\circ} \in F(E_B)$, this $F(f)$ corresponds to an arrow $F'(f) : F'(A) \to F'(B)$ in $\mathbf{X}'$.

Clearly, identity arrows in $\mathbf{X}$, corresponding to identity arrows in $\mathcal{G}(\mathbf{X})$ and mapped to identities in $\mathcal{G}(\mathbf{X}')$ under F, will be mapped to identities in $\mathbf{X}'$ under F'. We check that composition is preserved. Suppose that f and g are arrows whose composite gf exists in $\mathbf{X}$. Both g and f correspond, then, to arrows $g : \overline{g} \to \overline{g^\circ}$ and $f : \overline{f} \to \overline{f^\circ}$, respectively, in $\mathcal{G}(\mathbf{X})$. Notice that the composite gf does not necessarily exist in $\mathcal{G}(\mathbf{X})$, but that, since $\overline{g}, \overline{f^\circ} \in E_B$, the tensor $g \otimes f$ does and that this tensor product uniquely corresponds to gf by Proposition 2.8. By Proposition 2.9, then, $F(g \otimes f) = F(g) \otimes F(f)$ and, again by Lemma 2.8 and the definition of F', corresponds to $F'(g)F'(f)$. □

Construction 2.12. Given a top-heavy locally inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i\in I})$, define an inverse category $\left(\mathcal{I}(\mathbf{G}), \circ, \overline{(-)}\right)$ with the following data:

- Objects: The objects are the meet-semilattices M_i.

- Arrows: $\mathcal{I}(\mathbf{G})(M_1, M_2) = \{f : A_1 \to A_2 \text{ in } \mathbf{G} \,|\, A_1 \in M_1,\ A_2 \in M_2\}$. Note that every object of $\mathbf{G}$ is in some M_i, so that every arrow in $\mathbf{G}$ will be found in exactly one of these hom-sets since the M_i's are disjoint.
  - Composition: A composable pair of arrows $f : M_1 \to M_2$ and $g : M_2 \to M_3$ in $\mathcal{I}(\mathbf{G})$, corresponds to a pair of arrows $f : A_1 \to A_2$ and $g : A_2' \to A_3$ in $\mathbf{G}$ with $A_1 \in M_1$, $A_2, A_2' \in M_2$ and $A_3 \in M_3$. Since M_2 is a meet-semilattice, the meet $A_2 \wedge A_2'$ exists. We can therefore define the composite of f with g as $g \circ f = g \otimes f = [g\,|_*\, A_2 \wedge A_2'][A_2 \wedge A_2'{}_*|\, f]$. This composition is associative by Proposition 1.4.
  - Identities: For each object M_1, define $1_{M_1} : M_1 \to M_1$ to be $1_{\top_1} = \top_1 \to \top_1$ in $\mathbf{G}$. Let $f : M_1 \to M_2$ be an arrow corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$. Note that $[1_{\top_1}\,|_*\, A_1 \wedge \top_1] = 1_{A_1}$ by Proposition 1.5. Then
$$f \circ 1_{\top_1} = [f\,|_*\, A_1 \wedge \top_1] \bullet [A_1 \wedge \top_1{}_*|\, 1_{\top_1}] = [f\,|_*\, A_1] \bullet 1_{A_1} = f.$$
Similarly, $1_{\top_2} \circ f = f$.
  - Restrictions: Given an arrow $f : M_1 \to M_2$ corresponding to an arrow $f : A_1 \to A_2$ in $\mathbf{G}$, define $\overline{f} : M_1 \to M_1$ by $\overline{f} = 1_{A_1} : A_1 \to A_1$. Conditions (R.1) – (R.4) saying that $\mathcal{I}(\mathbf{G})$ is a restriction category follow readily from the fact that all restriction idempotents are identities on some object in $\mathbf{G}$ and that restrictions in an ordered groupoid are unique.
  - Partial Isomorphisms: For each arrow $f : M_1 \to M_2$, define $f^\circ : M_2 \to M_1$ as $f^{-1} : A_2 \to A_1$. To check that this is a restricted inverse, we check the required composites. First,
$$f \circ f^\circ = f \otimes f^\circ = [f\,|_*\, A_1 \wedge A_1] \bullet [A_1 \wedge A_1{}_*|\, f^{-1}] = f \bullet f^{-1} = 1_{A_2} = \overline{f^{-1}}.$$
Similarly, $f^\circ \circ f = \overline{f}$.

Proposition 2.13. *For each locally inductive functor $F : \mathbf{G} \to \mathbf{H}$, there exists a functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{H})$.*

Proof. We show that F induces a functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{H})$.

Given any object in $\mathcal{I}(\mathbf{G})$, a meet-semilattice M_1, define $\mathcal{I}(F)(M_1)$ to be *the* meet-semilattice M_1' such that $F(M_1) \subseteq M_1'$. Note that this assignment of M_1' to M_1 is unique since the M_i' are a partition of $\mathbf{H}_0$.

For any arrow $f : M_1 \to M_2$ in $\mathcal{I}(\mathbf{G})$ corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$, we define $\mathcal{I}(F)(f) = F(f) : F(A_1) \to F(A_2)$, an arrow $F(f) : F(M_1) \to F(M_2)$ in $\mathcal{I}(\mathbf{G}')$. That this assignment is functorial follows from the functoriality of F. □

Corollary 2.14. *Construction 2.12 is the object function of a functor $\mathcal{I} : \mathbf{TLIGrpd} \to \mathbf{ICat}$.* □

Proof. Let $\mathbf{G} \xrightarrow{F} \mathbf{G}' \xrightarrow{G} \mathbf{G}''$ be a composable pair of locally inductive functors. Then, on objects of $\mathcal{I}(\mathbf{G})$ (meet-semilattices forming the partition of $\mathbf{G_0}$),

$$\begin{aligned}\mathcal{I}(G)\mathcal{I}(F)(M) &= \mathcal{I}(G)(M'),\ \text{where } M' \text{ such that } FM \subseteq M'\\ &= M'',\ \text{where } M'' \text{ such that } M'' \supseteq G(M') = G(FM) = (GF)M\\ &= \mathcal{I}(GF)(M),\ \text{by the uniquess of } M'' \supseteq (GF)M.\end{aligned}$$

Equality of the functors $\mathcal{I}(GF)$ and $\mathcal{I}(G)\mathcal{I}(F)$ follows immediately. That $\mathcal{I}$ preserves identity functors follows from the observation that $\mathcal{I}(1_{\mathbf{G}})(M) = M$ for all objects M in $\mathcal{I}(\mathbf{G})$. □

Theorem 2.15. *The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence of categories,*

$$\mathbf{ICat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{TLIGrpd}$$

Proof. By Corollary 2.11, the functor $\mathcal{G}$ is fully faithful. We show now that $\mathcal{G}$ is essentially surjective by demonstrating a natural isomorphism $\mathcal{GI} \cong 1_{\mathbf{TLIGrpd}}$.

We start with a top-heavy locally inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i\in I})$ and we consider the composite $\mathcal{GI}(\mathbf{G})$. Recall that $\mathcal{I}(\mathbf{G})$ has as objects the meet-semilattices M_i and arrows of the form $f : M_1 \to M_2$, where $f : A_1 \to A_2$ is an arrow in $\mathbf{G}$ with $A_1 \in M_1$ and $A_2 \in M_2$. Further recall that every arrow in $\mathbf{G}$ is found exactly once in $\mathcal{I}(\mathbf{G})$. Note that for each object M_i,

$$E_{M_i} = \{\overline{f} : M_i \to M_i | f : M_i \to M_i\} = \{1_{A_i} | A_i \in M_i\} \cong M_i.$$

Then the locally inductive groupoid $\mathcal{GI}(\mathbf{G})$ contains the following data:

- Objects: $\coprod_{i\in I} E_{M_i} \cong \coprod_{i\in I} M_i = \mathbf{G}_0$.
- Arrows: For each $f : M_1 \to M_2$ in $\mathcal{I}(\mathbf{G})$ corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$, there is an arrow $f : \overline{f} \to \overline{f^\circ} = f : 1_{A_1} \to 1_{A_2} \cong f : A_1 \to A_2$ in $\mathcal{GI}(\mathbf{G})$. Since arrows of $\mathbf{G}$ are appearing exactly once in $\mathcal{I}(\mathbf{G})$, we have, then, that $(\mathcal{GI}(\mathbf{G}))_1 \cong \mathbf{G}_1$.
  - Composition: Given two composable arrows corresponding to $f : A_1 \to A_2$ and $g : A_2 \to A_3$ in $\mathcal{GI}(\mathbf{G})$, we have in $\mathcal{I}(\mathbf{G})$ that $g \circ f = g \otimes f = [g\,|_*\, A_2 \wedge A_2] \bullet [A_2 \wedge A_2\, {}_*|\, f] = g \bullet f$. Their composite, then, is
$$g \star f \text{ in } \mathcal{GI}(\mathbf{G}) = g \circ f \text{ in } \mathcal{I}(\mathbf{G}) = g \bullet f \text{ in } \mathbf{G}.$$
    That is, composition in $\mathcal{GI}(\mathbf{G})$ is the same as that in $\mathbf{G}$ up to isomorphism.
  - Restrictions: Given an arrow $f : 1_{A_1} \to 1_{A_2} \cong f : A_1 \to A_2$ and $A_1' \leq A_1$, we have that
$$\begin{aligned}(f\,|_*\, A_1') \text{ in } \mathcal{GI}(\mathbf{G}) \cong f \circ 1_{A_1'} \text{ in } \mathcal{I}(\mathbf{G}) &= f \otimes 1_{A_1'} \text{ in } \mathbf{G} \\ &= [f\,|_*\, A_1 \wedge A_1'] \bullet [A_1 \wedge A_1'\,|_*\, 1_{A_1'}] \\ &= [f\,|_*\, A_1'] \bullet 1_{A_1'} = [f\,|_*\, A_1'].\end{aligned}$$
    That is, the restrictions of the two ordered groupoids $\mathbf{G}$ and $\mathcal{GI}\mathbf{G}$ are the same up to isomorphism.

This description of $\mathcal{GI}(\mathbf{G})$ is written so that the isomorphism $\mathbf{G} \cong \mathcal{GI}(\mathbf{G})$ follows immediately. □

The definition of the functor $\mathcal{G}$ relies on the top-heavy property of a locally inductive groupoid $\mathcal{G}$ only when defining identities on the meet-semilattices partitioning $\mathbf{G}_0$. Similarly, the identities of an inverse category $\mathbf{X}$ are essential only as top elements of the meet-semilattices E_A. In other words, removing identities from an inverse category is equivalent to removing top elements from the meet-semilattices partitioning a locally inductive groupoid. As a result, the equivalence established in Theorem 2.15 generalizes immediately.

Corollary 2.16. *The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence*

$$\mathbf{ISCat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{LIGrpd},$$

where **ISCat** *is the category of inverse semicategories.* □

Since single-object inverse categories are precisely inverse semigroups with unit, it is clear that single-object inverse semicategories are precisely inverse semigroups. With inverse semicategories as multi-object inverse semigroups, we see that Theorem 1.9 – the equivalence between inductive groupoids and inverse semigroups – is then a corollary of Corollary of 2.16.

We will end this paper with a short discussion on a generalization of Theorem 2.15.

Recall that *prehomomorphisms* of inverse semigroups are functions between inverse semigroups satisfying $\phi(ab) \leq \phi(a)\phi(b)$. Theorem 1.9 can then be generalized to

Theorem 2.17 ([5], Theorem 8)**.** *The category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors.* □

Since the arrows of an inverse category are playing the part of "elements" in each of the "local inverse semigroups", a clear candidate for an inverse categorical analogue arises.

Definition 2.18. An *oplax functor* $F : \mathbf{X} \to \mathbf{X}'$ of inverse categories consists of the following data:

- for each object $A \in \mathbf{X}$, an object $F(A) \in \mathbf{X}'$;
- for each arrow $f : A \to B$, an arrow $F(f) : F(A) \to F(B)$ such that
  - for each composable pair $f : A \to B$ and $g : B \to C$ in $\mathbf{X}$, $F(gf) \leq F(g)F(f)$, and
  - for each object $A \in \mathbf{X}$, $F(1_A) \leq 1_{F(A)}$.

Clearly, since composition in $\mathcal{G}(\mathbf{X})$ is defined by composition in $\mathbf{X}$, any oplax functor $F : \mathbf{X} \to \mathbf{X}'$ between inverse categories induces an ordered functor $\mathcal{G}(F) : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{X}')$.

Suppose now that $F : \mathbf{G} \to \mathbf{G}'$ is an ordered functor between top-heavy locally inductive groupoids. Recall that composition in $\mathcal{I}(\mathbf{G})$ is defined by the tensor product in $\mathbf{G}$. Then

$$\begin{aligned} F(g \otimes f) &= F(g \,|_*\, \mathrm{dom}(g) \wedge \mathrm{cod}(f)) F(\mathrm{dom}(g) \wedge \mathrm{cod}(f) \,{}_*|\, f) \\ &= (Fg \,|_*\, F(\mathrm{dom}(g) \wedge \mathrm{cod}(f)))(F(\mathrm{dom}(g) \wedge \mathrm{cod}(f)) \,{}_*|\, Ff) \\ &\leq (Fg \,|_*\, F\mathrm{dom}(g) \wedge F\mathrm{cod}(f))(F\mathrm{dom}(g) \wedge F\mathrm{cod}(f) \,{}_*|\, Ff) \\ &= Fg \otimes Ff \end{aligned}$$

and thus F induces an oplax functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{G}')$. Specifically, since the identities in $\mathcal{I}(\mathbf{G})$ are the top elements of $\mathbf{G}$, $\mathcal{I}(F)$ is strict on identities.

The arguments in this paper can then be easily extended to prove the following.

Theorem 2.19. *The category of top-heavy locally inductive groupoids and ordered functors is equivalent to the category of inverse categories and oplax functors.* □

References

[1] J. R. B. Cockett and Stephen Lack. Restriction categories I: categories of partial maps. *Theoretical Computer Science*, 270:223–259, 2002.

[2] Christopher Hollings. The Ehresmann–Schein–Nambooripad Theorem and its successors. *European Journal of Pure and Applied Mathematics*, pages 1–39, 2009.

[3] Christopher Hollings. Extending the Ehresmann-Schein-Nambooripad theorem. *Semigroup Forum*, 80(3):453–476, 2010.

[4] M. V. Lawson. Semigroups and ordered categories. I. The reduced case. *J. Algebra*, 141(2):422–462, 1991.

[5] Mark V. Lawson. *Inverse Semigroups: The Theory of Partial Symmetries*. World Scientific Publishing Co., 1998.

[6] Markus Linckelmann. On inverse categories and transfer in cohomology. *Proceedings of the Edinburgh Mathematical Society. Series II*, 56(1):187–210, 2013.

[7] K.S.S Nampooripad. *Structure of Regular Semigroups*. PhD thesis, University of Kerala, 1973.

[8] K.S.S Nampooripad. Structure of regular semigroups, I. Fundamental regular semigroups. *Semigroup Forum*, 9(1):354–363, 1975.

[9] K.S.S Nampooripad. Structure of regular semigroups, II. The general case. *Semigroup Forum*, 9(1):364–371, 1975.

[10] G.B. Preston. *Some Problems in the Theory of Ideals*. PhD thesis, University of Oxford, 1953.

[11] B.M. Schein. On the theory of inverse semigroups and generalised groups. *American Mathematical Society Translations*, 2(113):89–122, 1979.