Restriction Category Perspectives of Partial Computation and Geometry

by

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Dedication

For Roori Lee, whom I *sure as shouting* will never forget.

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Abstract

This thesis introduces several structures based on Cockett and Lack's restriction categories which find applications in partial computation, geometry, topology and two-dimensional category theory.

As monads may be thought of as modelling computation, we introduce restriction monads as a candidate to model partial computation. These structures are defined in the hope that any dependencies on the restriction structure of a category can be abstracted instead into an endofunctor on it. For example, we prove that the data of a small restriction category can be encoded as a restriction monad in Span(**Set**). We introduce restriction bimodules (bimodules whose actions are partially defined) and prove that they are algebras for these monads.

The Ehresmann-Schein-Nambooripad theorem asserts an equivalence between the categories of inverse semigroups and that of certain groupoids. We prove that this equivalence can be extended to an equivalence between inverse categories (categories in which all arrows are partially invertible) and top-heavy locally inductive groupoids (categories in which all arrows are totally invertible, all arrows have a notion of restriction and corestriction, and the objects may be partitioned into meet-semilattices) in two different ways (using two different notions of morphism). For any join inverse category $\mathbf{X}$, we prove that the corresponding top-heavy locally inductive groupoid $\mathcal{G}(\mathbf{X})$ is locally localic and that each morphism in $\mathbf{X}$ gives an equivalence between the locales generated by the principal order ideals of its source and target. We then prove that $\mathcal{G}(\mathbf{X})$ can be naturally given the structure of an Ehresmann site, which then motivates our definition of ideally covering and ideally flat functors between Ehresmann sites that make the constructions of Lawson and Steinberg functorial.

Finally, we introduce double restriction categories (a double category equipped with two compatible restriction structures) and restriction bicategories (bicategories with a "weak" restriction operator). We show that the bicategory of restriction modules can be given the structure of a restriction bicategory and use these to organize restriction monads, restriction modules, monad morphisms, and module morphisms into a double restriction category.

List of Abbreviations and Symbols Used

Notation	Description
$\square$	End of proof.
$\blacktriangle$	End of example.
$\diamondsuit$	End of construction or definition.
Par	The category of sets and partial functions.
$\mathrm{Par}(\mathbf{X}, \mathcal{M})$	The category of partial maps in $\mathbf{X}$ with respect to a stable system of monics $\mathcal{M}$,
$\overline{f}$	The restriction idempotent of f.
f°	The restricted or pseudo inverse of f with respect to $\circ$.
$\mathbf{X}^{\mathbf{Y}}$	The category whose objects are functors $\mathbf{Y} \to \mathbf{X}$ and whose arrows are natural transformations.
$\mathrm{Span}(\mathbf{Set})$	The bicategory whose 0-cells are sets, whose 1-cells are spans and whose 2-cells are span morphisms.
$R(\mathbf{X})$	The restriction monad in $\mathrm{Span}(\mathbf{Set})$ corresponding to the small restriction category $\mathbf{X}$.
$\mathcal{G}(\mathbf{X})$	The (top-heavy locally) inductive groupoid corresponding to the inverse semigroup (category) $\mathbf{X}$.
$\mathcal{I}(\mathbf{G})$	The inverse semigroup (category) corresponding to the (top-heavy locally) inductive groupoid $\mathbf{G}$.
oGrpd	The category of ordered groupoids and ordered functors.
iGrpd	The category of inductive groupoids and inductive functors.
iSgp	The category of inverse semigroups and homomorphisms.
iCat	The category of inverse categories and functors.

Notation	Description
tliGrpd	The category of top-heavy locally inductive groupoids and inductive functors.
E_A	The set of restriction idempotents whose source (and target) are A.
liGrpd	The category of locally inductive groupoids and inductive functors.
isCat	The category of inverse semicategories and functors.
jiCat	The category of join inverse categories and join-preserving functors.
$\downarrow \overline{f}$	The principal order ideal of the restriction idempotent $\overline{f}$.
lcCat	The category of left-cancellative categories and functors.
$(\mathbf{G}, T)$	An Ehresmann site, a groupoid $\mathbf{G}$ equipped with an Ehresmann topology T.
$\mathcal{G}_C\mathbf{C}$	The ordered groupoid associated to the left-cancellative category $\mathbf{C}$.
$\mathcal{L}_C(\mathbf{G})$	The left-cancellative category associated to the groupoid $\mathbf{G}$.
srModule($\mathbf{X}$)	The bicategory of supported range modules in $\mathbf{X}$.
$\mathbb{R}$Module($\mathbf{X}$)	The double category of restriction modules in $\mathbf{X}$.

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Chapter 1: Introduction

Consider the real-valued functions $f(x) = x+2$ and $g(x) = \sqrt{x}$. We teach our students to identify the domain of f to be the entirety of the real numbers and the domain of g to be only non-negative real numbers. In effect, we teach our students to identify the (largest) subset of the real numbers on which a real-valued function is defined. In this sense we could say that f is *totally* defined on $\mathbb{R}$ while g is only *partially* so. To compute the composite $g \circ f$ of these functions, we teach students to again identify the domain on which the composite is defined by intersecting the image of f and the domain of g. This process can be described structurally by considering f and g as *partial functions* $f, g : \mathbb{R} \,{-}\!\!\bullet\!\!{\rightarrow}\, \mathbb{R}$.

Definition 1.0.1. A partial function $f : A \,{-}\!\!\bullet\!\!{\rightarrow}\, B$ (of sets) is a pair of (total, or fully defined) functions

$$A \xleftarrow{\;i\;} D_f \xrightarrow{\;f\;} B$$

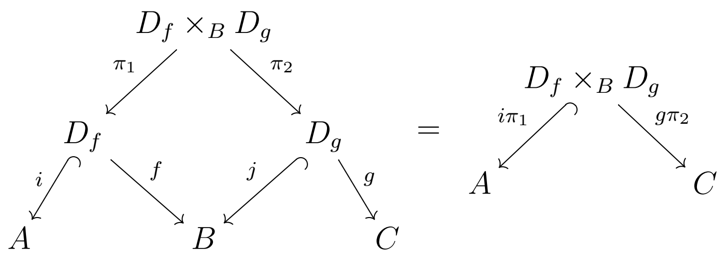
where

- i is just inclusion of D_f into A, and
- we think of D_f as the domain of definedness of f. ◇

One composes two partial functions $A \overset{f}{\,{-}\!\!\bullet\!\!{\rightarrow}\,} B \overset{g}{\,{-}\!\!\bullet\!\!{\rightarrow}\,} C$

$$A \xleftarrow{\;i\;} D_f \xrightarrow{\;f\;} B \xleftarrow{\;j\;} D_g \xrightarrow{\;g\;} C$$

by first taking the pullback and then composing along the legs:



The chosen pullback, explicitly, is

$$\begin{aligned} D_f \times_B D_g &= \{(a,b) \in D_f \times D_g : b = f(a)\} \\ &= \mathrm{Im}(f) \cap D_g \end{aligned}$$

In other words, this is exactly how one is taught in precalculus to "find the domain" of the composite $(g \circ f)(x)$. With a structural interpretation of composing partial functions, we can then organize these data into a category **Par** of sets and partial functions, with composition as described above.

It may be interesting to model partially defined "functions" in other categories whose objects may not be sets. Pullbacks may not exist or a category may not even have enough objects (a category could, indeed, have only one) to talk about "subobjects". We solve this in a category by phrasing our definitions in terms of the arrows and their composites. We can solve the object problem if we can define "domain of definedness" in terms of other partial functions (more generally, arrows), rather than relying on sets (more generally, objects). We then will want to impose some axioms about how this domain of definedness behaves with other arrows.

In **Par**, for each partial function $f : A \!-\!\!\bullet\!\!\rightarrow B$, define a new partial function

$\overline{f} : A \longrightarrow\!\!\!\!\!\bullet\!\!\!\rightarrow A$ by

$$\overline{f}(x) = \begin{cases} x, & \text{if } x \in D_f \\ \text{Not defined}, & \text{otherwise} \end{cases}$$

This translates a set-based definition into a partial function-based one so that we can use it in an arbitrary category.

Restriction Categories

Categorical modelling of partial maps is well studied [1, 21, 22, 31, 34]. The algebraic theory of partially defined functions, using a partial identity on the domain to encode the domain of definedness for the partial functions, can be described nicely by Cockett and Lack's notion of restriction category, which allows us to very naturally reason about partial maps in an equational manner.

This thesis uses restriction categories as a basis for developing data structures used in the study of partial computation, geometry, topology and higher dimensional restriction categories.

Definition. *A restriction structure on a category* $\mathbf{X}$ *is an assignment of an arrow* $\overline{f} : A \to A$ *to each arrow* $f : A \to B$ *in* $\mathbf{X}$ *satisfying the following four conditions:*

(R.1) For all maps f, $f\,\overline{f} = f$.

(R.2) For all maps $f : A \to B$ *and* $g : A \to B'$, $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

(R.3) For all maps $f : A \to B$ *and* $g : A \to B'$, $\overline{\overline{g}\,\overline{f}} = \overline{g}\,\overline{f}$.

(R.4) For all maps $f : B \to A$ *and* $g : A \to B'$, $\overline{g}\,f = f\,\overline{gf}$.

A category equipped with a restriction structure is called a restriction category*.*

Definition. *A* restriction functor $F : \mathbf{X} \to \mathbf{Y}$ *between restriction categories is a functor which preserves the restriction idempotents;* $F\left(\overline{f}\right) = \overline{F(f)}$ *for all* $f \in \mathbf{X}_1$*.*

Chapter 2 includes a more detailed review of restriction categories.

Partial Computation

Chapter 3 introduces *restriction monads*, a structure encoding partially defined abstract computations which is analogous to the encoding of (totally defined) abstract computations in the language of monads in [33]. Thinking of a monad T in a bicategory $\mathcal{B}$ (with an object E which will in some way "choose elements") as playing the role of a category, whose composition is defined by μ and whose units are given by η, a restriction structure on T is a family of functions $\rho_{A,B}$ indexed by certain 1-cells in $\mathcal{B}$ of the form $A, B : E \to X$ (which are playing the role of "E-elements of X"). This family of functions satisfies four conditions analogous to (R.1) through (R.4). Indeed, Proposition 3.1.2 shows that these axioms are exactly those for which small restriction categories are in correspondence with restriction monads in Span(**Set**).

A small category also corresponds to a monad in **Set**-Mat (sets and matrices – full details in Chapter 5) encoding the data as a category *enriched in* **Set**. This correspondence extends to restriction monads, given by Proposition 3.1.3.

We then define algebras for restriction monads by equipping ordinary algebras with a restriction structure. As right **X**-modules are the algebras for the ordinary monad corresponding to a small category **X**, Theorem 3.2.2 states that the algebras for restriction monads in the bicategory Span(**Set**) are certain *restriction (bi)modules*:

a restriction bimodule φ is a bimodule between restriction categories $\mathbf{X}$ and $\mathbf{Y}$ so that each element $\alpha \in \varphi(y, x)$ can be assigned a restriction idempotent $\overline{\alpha} : x \to x$ of $\mathbf{X}$ which behaves as a restriction structure with respect to the actions of $\mathbf{X}$ and $\mathbf{Y}$ on φ.

In Chapter 5, restriction bimodules provide a motivating example of a two dimensional restriction structure (see the subsection below).

Geometry and Topology

Chapter 4 introduces *top-heavy locally inductive groupoids*, groupoids whose objects can be partitioned into meet-semilattices and whose arrows have a notion of restriction and corestriction. Our motivation for these groupoids comes from the Ehresmann-Schein-Nambooripad Theorem, which asserts a correspondence between inductive groupoids (ordered groupoids whose objects form one meet-semilattice) and inverse semigroups, stated in full in Theorem 4.1.9.

Partitioning a top-heavy locally inductive groupoid into meet-semilattices, we prove that these groupoids are the suitable "multi-object" version of inductive groupoids, in the sense that Theorem 4.1.9 can be generalized to inverse categories. For any inverse category $\mathbf{X}$, we construct a top-heavy locally inductive groupoid $\mathcal{G}(\mathbf{X})$ (Proposition 4.2.7) whose objects are the restriction idempotents and whose arrows are the arrows of $\mathbf{X}$ (with their sources and targets suitably defined). We prove that this construction is functorial, fully faithful and essentially surjective in Theorem 4.2.15.

The object function of the functors forming this equivalence was independently discovered by Lawson in an unpublished preprint, which was communicated by private communication after our submission for publication. Lawson's proof, in addition to omitting the arrow function of the functors, takes advantage of the units of an inverse category. Our proof, taking advantage rather of the partition has the added benefit that the Ehresmann-Schein-Nambooripad Theorem is a corollary (since single-object inverse semicategories are inverse semigroups).

The category of top-heavy locally inductive groupoids above has inductive functors (functors which preserve any meets that exist). If we replace these functors with order preserving functors, our proof yields a further generalization (Theorem 4.2.19) of the Ehresmann-Schein-Nambooripad Theorem: the category of top-heavy locally inductive groupoids and ordered functors is equivalent to the category of inverse categories and oplax functors.

Two arrows in a restriction category are *compatible* [8] if $f\,\overline{g} = g\,\overline{f}$; that is, if f and g agree where they are both defined. For each x in the intersection of the domains of two partial functions (of sets) f and g, it must be the case that $f(x) = g(x)$, lest x be mapped to two different places and the join will not be a partial function. In general, the join of two arrows in a restriction category will be possible only if the two arrows are compatible. Restriction categories with joins were used in [8] as a setting for constructing manifolds. We show that the top-heavy locally inductive groupoid associated to a join *inverse* category is locally localic; for each object of $\mathcal{G}(\mathbf{X})$, its *principal order ideal* (Definition 4.3.5) is a locale (Theorem 4.3.7). In addition, Proposition 4.3.8 asserts that locale homomorphisms in this groupoid are given by the morphisms of the inverse category. Corollaries 4.3.9 and 4.3.11 construct covariant and contravariant locale-valued functors defined on $\mathcal{G}(\mathbf{X})$. Finally, we prove in Theorem 4.3.12 that these functors give an equivalence between the locales given by the principal order ideals.

An Ehresmann site [28] is an ordered groupoid equipped with an Ehresmann topology: for each object, a family of order ideals of that object satisfying some conditions reminiscent of a Grothendieck topology on a category. The locally localic structure of $\mathcal{G}(\mathbf{X})$ allows us to equip it with a natural Ehresmann topology, as defined in Theorem 4.3.18.

Theorems 4.4.8(d) and 4.4.11(d) establish a correspondence between Grothendieck topologies on a left cancellative category and Ehresmann topologies on ordered groupoids. Using these correspondences, Definition 4.4.13 gives the appropriate notion of morphisms between Ehresmann sites, which can be constructed from morphisms of left-cancellative categories using the construction given in Theorem 4.4.15.

Two Dimensional Restriction Categories

Chapter 5 introduces *double restriction categories* and *restriction bicategories*. Double restriction categories are data structures with two compatible restriction structures, while restriction bicategories are such that the restriction structure on the 1-cells extends functorially to its 2-cells (with the restriction axioms holding up to coherent isomorphisms).

We first define a restriction structure in the language of internal categories, so that we can take restriction categories internal to the category of restriction categories; a restriction category (in **Set**) contains the following data:

$$\mathbf{X}_1 \, {}_t\!\times_s \mathbf{X}_1 \xrightarrow{\;c\;} \mathbf{X}_1 \overset{s}{\underset{t}{\rightrightarrows}} \mathbf{X}_0, \qquad \mathbf{X}_0 \xrightarrow{\;u\;} \mathbf{X}_1, \qquad r\colon \mathbf{X}_1 \to \mathbf{X}_1$$

The full definition can be found in Definition 5.1.1, but the main takeaway is that the morphism $r : \mathbf{X}_1 \to \mathbf{X}_1$ is thought of as the restriction operator. As such, in addition to the usual unit and associativity axioms, a restriction category internal to **Set** must satisfy the (suitably re-stated) restriction axioms, as well as an axiom forcing the source of $r(f)$ to be the source of f.

We may also take such structures internal to any category with pullbacks over s and t. In particular, a restriction category internal to **Set** is a small restriction category with $\overline{f} = r(f)$. As such, a restriction category internal to **Set** satisfies equations corresponding to facts about restriction categories (cf. Proposition 2.0.3).

Having a suitable diagrammatic description of restriction categories, we define a double restriction category to be a restriction category internal to the category of restriction categories (with restriction functors). One can then verify that this definition is equivalent to defining a double restriction category as a double category – which is expressed in terms of objects, vertical arrows, horizontal arrows and double cells – whose horizontal and vertical categories are equipped with a vertical restriction structure $\overline{(-)}$ and horizontal restriction structure $\widetilde{(-)}$ such that $\overline{(-)} \circ \widetilde{(-)} = \widetilde{(-)} \circ \overline{(-)}$. This is detailed in Definition 5.1.5.

Example 5.1.11 shows that the double categorical span construction can be generalized to restriction categories where pullbacks may also not be completely defined, via a suitable subcategory of partial maps.

Finally, we recall that certain restriction bimodules are the algebras for the restriction monad in Span(**Set**) corresponding to the small restriction category $\mathbf{X}$. We show that the composite of two restriction bimodules (defined by the standard coequalizer diagram) is again a restriction bimodule. This composition is associative only up to invertible 2-cell. In addition, we show that any restriction bimodule φ can be assigned a new restriction bimodule $\overline{\varphi}$ and that this assignment satisfies the restriction axioms up to invertible 2-cell.

As such, we consider the restriction bimodules as a motivating example of a restriction bicategory, which is a like a restriction category whose restriction axioms hold up to canonical invertible 2-cell.

Tying all of these structures together, we define two double categories.

	$\mathbb{R}$Module(**rCat**)	$\mathbb{R}$Mod(Span(**Set**))
Objects	Rest. Cats.	Rest. Monads in Span(**Set**)
Vertical Arrows	Rest. Functors	Monad Morphisms
Horizontal Arrows	Rest. Modules	Algebras
Double Cells	Equivariant Maps	Equivariant Maps

$$\begin{array}{ccc} \mathbf{X} & \overset{M}{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;\;} & \mathbf{X}' \\ {\scriptstyle F}\downarrow & \alpha & \downarrow{\scriptstyle F'} \\ \mathbf{Y} & \underset{M'}{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;\;} & \mathbf{Y}' \end{array} \qquad\qquad \begin{array}{ccc} T & \overset{A}{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;\;} & T' \\ {\scriptstyle F}\downarrow & \alpha & \downarrow{\scriptstyle F'} \\ N & \underset{B}{\longrightarrow\!\!\!\!\!\!\!\!\mid\;\;\;\;} & N' \end{array}$$

$\mathbb{R}$Module(**rCat**)) $\qquad$ $\mathbb{R}$Mod(Span(**Set**)

Analogous to range categories [7], a range module is a restriction bimodule $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\!\mid\;\;\; \mathbf{Y}$ which encodes range in addition to domain. A supported range module is a module whose range structure is also a co-restriction structure.

Consider the double category $s\mathbb{R}$Module(**iCat**), with supported range modules between inverse categories as horizontal arrows, with functors as vertical arrows and with restriction module morphisms as double cells. Similarly, we define the double category $s\mathbb{R}$Module(**tliGrpd**). We then conjecture the existence of an extension of the functor $\mathcal{I} : \mathbf{tliGrpd} \to \mathbf{iCat}$ to a biequivalence $\mathcal{I} : \text{srModule}(\mathbf{tliGrpd}) \to \text{srModule}(\mathbf{iCat})$ which further extends to an equivalence of the double categories $s\mathbb{R}\text{Module}(\mathbf{iCat})$ and $s\mathbb{R}\text{Module}(\mathbf{tliGrpd})$.

Chapter 2: Preliminaries

Restriction Categories

Definition 2.0.1 ([11])**.** A restriction structure on a category $\mathbf{X}$ is an assignment of an arrow $\overline{f} : A \to A$ to each arrow $f : A \to B$ in $\mathbf{X}$ satisfying the following four conditions:

(R.1) For all maps f, $f\,\overline{f} = f$.

(R.2) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$.

(R.3) For all maps $f : A \to B$ and $g : A \to B'$, $\overline{g\,\overline{f}} = \overline{g}\,\overline{f}$.

(R.4) For all maps $f : B \to A$ and $g : A \to B'$, $\overline{g}\,f = f\,\overline{g f}$.

A category equipped with a restriction structure is called a *restriction category.* $\diamondsuit$

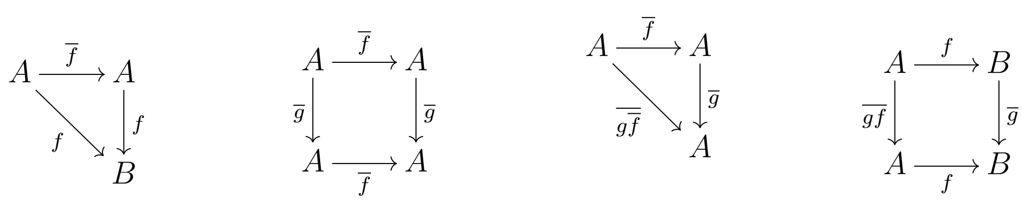


Figure 2.1: The axioms of a restriction category, diagrammatically.

Definition ([11])**.** *A* restriction functor $F : \mathbf{X} \to \mathbf{Y}$ *between restriction categories is a functor which preserves the restriction idempotents;* $F\left(\overline{f}\right) = \overline{F(f)}$ *for all* $f \in \mathbf{X}_1$.

Example 2.0.2. Examples of restriction categories:

(a) **Par** as defined above is the prototypical example of a restriction category. The axioms (R.1) – (R.4) required for **Par** to be a restriction category are easily verified. We can interpret expressions such as $f\,\overline{g}$ as "f restricted to where g is defined".

(b) Let $\mathbf{C}$ be an ordinary category equipped with a stable system $\mathcal{M}$ of monics (all details of this example can be found in [11]). Define a category $\mathrm{Par}(\mathbf{C}, \mathcal{M})$ with the following data:

- Objects: Same objects as $\mathbf{C}$.
- Arrows: Isomorphism classes of spans

$$X \xleftarrow{\;i\;} D \xrightarrow{\;f\;} Y,$$

  where $i \in \mathcal{M}$. We will sometimes denote such an arrow (actually, its isomorphism class) as (i, f).

- Composition: Composition is given by pullback.
- Restrictions: Given any arrow (i, f), the assignment $\overline{(i, f)} = (i, i)$ defines a restriction structure on $\mathrm{Par}(\mathbf{C}, \mathcal{M})$. ▲

The next lemma lists some useful identities that will be used, without reference, to make calculations in restriction categories.

Lemma 2.0.3 ([11]). *If* $\mathbf{X}$ *is a restriction category, then:*

(i) $\overline{f}$ *is idempotent;*

(ii) $\overline{f}\,\overline{gf} = \overline{gf}$;

(iii) $\overline{\overline{g}f} = \overline{gf}$;

(iv) $\overline{\overline{f}} = \overline{f}$;

(v) $\overline{\overline{g}\,\overline{f}} = \overline{g}\,\overline{f}$;

(vi) if f *is monic, then* $\overline{f} = 1$;

(vii) $f\overline{g} = f$ *implies* $\overline{f} = \overline{f}\,\overline{g}$.

Note. A restriction category $\mathbf{X}$ has a natural, locally partially ordered 2-category structure: for any two parallel arrows $f, g : C \to D$ in $\mathbf{X}$, we define a partial order by $f \leq g$ if and only if $f = g\overline{f}$. Notice that if $f \leq g$, then

$$\overline{g}\,\overline{\overline{f}} = \overline{g}\,\overline{f} = \overline{\overline{g}f} = \overline{f}$$

and thus $\overline{f} \leq \overline{g}$.

Proposition 2.0.4. *Suppose that* a, A, b *and* B *are arrows in a restriction category* $\mathbf{X}$ *with* $a \leq A$ *and* $b \leq B$*. If the composites* ab *and* AB *exist, then* $ab \leq AB$*.*

Proof. Suppose that a, A, b and B are arrows in $\mathbf{X}$ with $a \leq A$, $b \leq B$ and such that the composites ab and AB exist. Then

$$AB\overline{ab} = AB\overline{b}\,\overline{ab} = Ab\overline{ab} = A\overline{a}b = ab$$

and thus $ab \leq AB$. □

Definition 2.0.5. A map f in a restriction category $\mathbf{X}$ is called *total* whenever $\overline{f} = 1$. ♢

Lemma 2.0.6 ([11])**.** *If* $\mathbf{X}$ *is a restriction category, then:*

(i) *every monomorphism is total;*

(ii) *if* f *and* g *are total, then* gf *is total;*

(iii) *if* gf *is total, then* f *is total;*

(iv) *the total maps form a subcategory, denoted* $\mathrm{Tot}(\mathbf{X})$.

Definition 2.0.7. A morphism $F : \mathbf{X} \to \mathbf{Y}$ of restriction categories (a *restriction functor*) is a functor such that $F\left(\overline{f}\right) = \overline{F(f)}$ for each $f \in X_1$. ♢

Inverse Categories

As groupoids are for groups, we will use a structure describing multi-object inverse semigroups. Inverse semigroups with units are exactly single-object inverse categories, so it seems that inverse (semi)categories could be appropriate for such a role.

Definition 2.0.8 ([23])**.** A category $\mathbf{X}$ is said to be an *inverse category* whenever, for each arrow $f : A \to B$ in $\mathbf{X}$, there exists a unique $f^\circ : B \to A$ in $\mathbf{X}$ such that $f \circ f^\circ \circ f = f$ and $f^\circ \circ f \circ f^\circ = f^\circ$. ♢

Definition 2.0.9. A map f in a restriction category $\mathbf{X}$ is called a *restricted isomorphism* whenever there exists a map g – called a *restricted inverse* of f – such that $gf = \overline{f}$ and $fg = \overline{g}$. ♢

Following from the commutation of idempotents (Restriction Category Axiom 2.0.1), we have the following property of restricted isomorphisms:

Theorem 2.0.10 (Lemma 2.18(vii), [11])**.** *If f is a restricted isomorphism, then its restricted inverse is necessarily unique.*

Note. If a category $\mathbf{X}$ has the *property* of being an inverse category, one can define a restriction *structure* on $\mathbf{X}$ by defining $\overline{f} = f^\circ f$. Indeed, with this restriction structure, every arrow in $\mathbf{X}$ is a restricted isomorphism and the restricted inverse of an arrow f is exactly f°. This justifies the following notation and definition.

Notation. Given a map f in a restriction category $\mathbf{X}$, we denote its restricted inverse (if it exists) by f°.

Definition 2.0.11. A restriction category $\mathbf{X}$ is called an *inverse category*, whenever every map f is a restricted isomorphism. ♢

Example 2.0.12. Some inverse categories:

(a) The category of sets and partial bijections.

(b) Any inverse semigroup with unit is a single-object inverse category.

(c) Any groupoid is an inverse category with all arrows total. ▲

Lemma 2.0.13 ([11])**.** *If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, then F preserves*

(i) total maps,

(ii) restriction idempotents,

(iii) restricted sections and

(iv) restricted isomorphisms.

Note. Any functor between inverse categories is a restriction functor preserving restricted isomorphisms. This follows from the restriction structure and restricted isomorphisms being defined as specific composites. We will therefore omit the words "inverse" and "restriction" when speaking of functors between inverse categories.

As expected, restriction idempotents are their own restricted inverse.

Proposition 2.0.14. *In an inverse category,* $\left(\overline{f}\right)^\circ = \overline{f}$ *for all arrows* f*.*

Proof. Since all arrows in an inverse category are restricted isomorphisms,

$$\left(\overline{f}\right)^\circ = (f^\circ f)^\circ = f^\circ (f^\circ)^\circ = f^\circ f = \overline{f}.$$

□

It is clear that inverse categories, interpreted as restriction categories in Definition 2.0.11, are exactly the same as inverse categories interpreted as multi-object inverse semigroups in Definition 2.0.8. In this thesis, we choose to think in terms of restriction categories for two reasons: firstly, the choice of notation in restriction categories facilitates calculations. Secondly, we prefer to think of (finite) inverse semigroups as collections of partial automorphisms on a (finite) set whose idempotents are partial identities – inverse categories in terms of restriction categories explicitly make use of this intuition.

Notation. We denote the category of inverse categories and functors by **iCat**.

Proposition 2.0.15. *If* $\mathbf{X}$ *and* $\mathbf{Y}$ *are inverse categories, then* $\mathbf{X}^{\mathbf{Y}}$ *(the category whose objects are functors* $\mathbf{Y} \to \mathbf{X}$ *and whose arrows are natural transformations) is an inverse category.*

Proof. Consider any natural transformation $\alpha : F \Rightarrow G : \mathbf{Y} \to \mathbf{X}$. For any arrow $f : A \to B$ in $\mathbf{Y}$, that $(-)^\circ$ is an involution together with the naturality of α applied to f° imply that

$$Ff.\alpha_A^\circ = (\alpha_A.Ff^\circ)^\circ = (Gf^\circ.\alpha_B)^\circ = \alpha_B^\circ.Gf,$$

or that the following diagram commutes:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \alpha_A^\circ}\uparrow & & \uparrow{\scriptstyle \alpha_B^\circ} \\ GA & \xrightarrow[Gf]{} & GB \end{array}$$

That is, there is a natural transformation $\alpha^\circ : G \Rightarrow F$ defined by $(\alpha^\circ)_A = \alpha_A^\circ$ for all objects $A \in \mathbf{Y}$. It follows immediately, then, that $\alpha^\circ\alpha\alpha^\circ = \alpha$ and $\alpha\alpha^\circ\alpha = \alpha$, and that α° is unique with respect to this property. □

Chapter 3: Partial Computation

Categorical structures as abstractions of computation have been well studied; for example, the effective topos [22, 34], categories of domains [1], total combinatory algebras [31], and Cartesian closed categories (as equivalent to lambda theories) [21].

A commonality of these models is that all computations are assumed to be completely defined, or are total. When programming, however, a computation (or function, method, program, etc.) takes a (perhaps empty) set of inputs and produces some observable effect (for example, the square of an integer or changing the brightness of a laptop screen). This notion of function often differs from that of a pure mathematical function:

- The function `int.random(m,n)` (returns a random integer between m and n) is non-determined in the sense that its output is not determined completely by its inputs.

- The function `open("file")` (opens the file whose name is given by a character string and returns it as an input/output stream) is not defined on the entirety of its domain (all character strings), failing if a file by that name does not exist.

That is, ordinary and everyday computations need not be defined on all inputs and the total models are thus insufficient. Restriction categories have since been used to unify these concepts; so-called Turing categories are classifying categories for partial combinatory logics [6] and invertible computation has been treated using Cartesian inverse categories [18].

Our preferred approach to modelling partial computations stems from the following observation: though many functions are not honest mathematical functions, we still use them as if they are. Specifically, we commonly compose functions (for example, open a file and then pass that output through a text parser and then close the file). As humans, we know that such composites can break down if, along the way, one of the functions fails. Also as humans, we must then use different methods of error-catching which are provided by our language of choice (for example, nested `if then` or `try do` statements). In addition to the code produced this way being cumbersome and unsightly, the programmer cannot possibly foresee every possible way in which a complicated algorithm can fail (or, practically, every possible way that a student can enter an answer in an online assignment).

Composing such functions in a way that reduces programming errors and maintenance time, while still being readable, is facilitated by separating the *types* of a language from the *computations* of those types. For example, consider a language whose types are sets (for example, `int`, `char`, etc.).

- For a set A, the power set $\mathcal{P}(A)$ can model non-determinism. A non-deterministic function $f : A \to B$ can be interpreted as an honest function $f : A \to \mathcal{P}(B)$, which sends an element $a \in A$ to a subset of B containing all of its possible outputs.

- Let E be a set of error messages (strings). Then for a set A, the set $A \coprod E$ (E is a set of error messages) can be used to model exception handling. If $f : A \to B$ is a possibly failing function, an element $a \in A$ will be mapped to either some element of B if no error occurs, or some error message in E otherwise; that is, f can be interpreted as an honest function $f : A \to B \coprod E$.

At first glance, it seems that such honest functions are not all too useful; since the domains and codomains of these functions no longer match, we can not compose them as functions. However, this does not pose a problem in practice: in the case of non-determinism, we can take the union of all subsets in the image of f to get a subset of B and in the exception handling, we can `break` the composite at the first sight of an error.

In general, an assignment $A \mapsto T(A)$ (for example, $T(A) = \mathcal{P}(A)$ or $T(A) = A \coprod E$) can be thought of as a notion of computation [33] (or a structured type). The functions $f : A \to B$ in a language are then *interpreted* as honest functions $f_T : A \to T(B)$ (we call f_T the T-interpretation of f) and T must be defined so that for each pair $f : A \to B$ and $g : B \to C$, their interpretations $f_T : A \to T(B)$ and $g_T : B \to T(C)$ are composable.

If T is a monad, these interpretations can be collected into a category whose objects are the types and whose arrows are the T-interpretations of functions. Recall that such a category is called the *Kleisli category* of T. In many functional programming languages (for example, Haskell), it is these Kleisli structures that are called `monads`. In this section, we propose the use of monads equipped with a restriction structure as a new model of partial computation.

3.1 Restriction Monads

This section introduces restriction monads, a so-called theory of partial computation; we equip a monad (T, η, μ) with sufficient structure so that we may interpret T as having a restriction structure with respect to μ. The advantage of this approach is that computations involving monads are strictly typed.

A small category may be interpreted as a monad in two different bicategories, each offering a different perspective of what a category "is":

1. A small category corresponds to a monad in Span(**Set**), encoding the data as a category *internal to* **Set**. In general, if **C** is a category with all pullbacks over the source and target maps, then a monad in Span(**C**) is a category internal to **C**.

2. A small category also corresponds to a monad in **Set**-Mat, encoding the data as a category *enriched in* **Set**. In general, if $\mathcal{V}$ is a monoidal category with small sums (over which its tensor product distributes from both sides), then a monad in $\mathcal{V}$-Mat is a category enriched in $\mathcal{V}$.

Given the appropriate definition of a restriction monad, one would expect that the correspondence between small categories and monads in Span(**Set**) would generalize to a correspondence between small *restriction* categories and *restriction* monads in Span(**Set**). Similarly, one would expect that small restriction categories correspond to restriction monads in **Set**-Mat. Our primary motivation for defining a restriction monad comes from the internal case, since we will use internal restriction categories to define double restriction categories in Chapter 5.

With this in mind, we will now explore the structure required to give a monad (T, η, μ) on $\mathbf{X}_0$ in Span(**Set**) a "restriction structure"; that is, a structure which translates to a restriction structure on its corresponding category. Recall that such a monad is a span of the form $\mathbf{X}_0 \xleftarrow{s} \mathbf{X}_1 \xrightarrow{t} \mathbf{X}_0$. The corresponding category **X** has objects $\mathbf{X}_0$, arrows $\mathbf{X}_1$, identities defined by η and composition defined by μ.

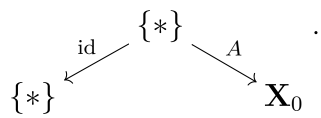
A naïve approach to equipping T with a restriction structure is to define a 2-cell

$\rho : T \Rightarrow T$ playing the part of the restriction operator. A sensible choice, but we immediately stumble when trying to write down an axiom corresponding to (R.1) of restriction categories. We know that it must conclude with μ, that ρ must be somewhere applied, and that the result of this μ-application must be the identity on T. That is, the composite

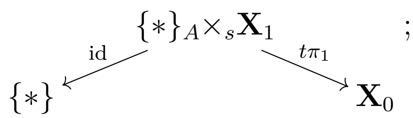
$$T \longrightarrow \cdots \longrightarrow T^2 \xrightarrow{T\rho} T^2 \xrightarrow{\mu} T$$

must be the identity. Attempting to fill in this diagram, we run into problems: how do we get from T to T^2 while adding non-trivial information (that is, without using η)? This approach also does not encode the primitive intuition of a restriction structure, an endomorphism on the source of an arrow which captures its definedness. With no way of encoding "endomorphism" or even "source", we must then impose these as additional structure: we require additional 1-cells D encoding "source" and E encoding "endomorphism". Naturally, we then require 2-cells to allow us to manipulate them. Though possible to define such structures so that a restriction monad in Span(**Set**) can be constructed from any small restriction category, the layering of the extra data D and E *on top* of T breaks the correspondence (an arbitrary restriction monad in Span(**Set**) does not uniquely determine a small restriction category). Instead, we will define restriction structure *within* the monad; we will assume that $\mathcal{B}$ has an object E which in some senses allows to to "choose elements". We will then define the restriction structure locally within the hom-category $\mathcal{B}(E, x)$. In Span(**Set**), E turns out to be a(ny) single element set.

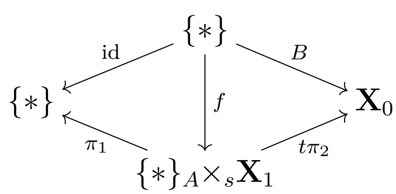
Suppose that $\mathbf{X}$ is a small restriction category. For each element A of $\mathbf{X}_0$, we can define a span $\vec{A} : \{*\} \longmapsto \mathbf{X}_0$ by



Composing such a span with T, then is of the form



that is, the composite $T\vec{A}$ contains as data all arrows of $\mathbf{X}$ with source A. Given another object $B \in \mathbf{X}_0$, a span morphism $f : \vec{B} \longmapsto T\vec{A}$, of the form



is therefore equivalent to the choice of an arrow f in $\mathbf{X}$ whose source is A and whose target is B; the hom-set $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ may be identified with the hom-set $\mathbf{X}(A, B)$. Such an identification seems, at first, a needlessly complicated workaround to accessing the elements of T. However, this identification allows us to define the restriction operator ρ as a family of set functions

$$\rho_{A,B} : \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \to \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

of arrows $f : A \to B$ to arrows $\rho(f) : A \to A$, which has the immediate advantage of all activity happening within the same hom-category $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)$ and eliminating our typing problems. Because the hom-categories of a bicategory are hom-*categories*, we can take advantage of the familiar diagonal $\Delta_X : X \to X \times X$ and canonical flip $\tau_{X,Y} : X \times Y \cong Y \times X$ afforded by the category of sets.

Identifying $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we must therefore consider how to "compose" elements of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}).$$

Noting that such an element is of the form

$$\begin{array}{ll} C \longrightarrow\!\!\!\!\!\!\mapsto TB & \\ & B \longrightarrow\!\!\!\!\!\!\mapsto TA, \end{array}$$

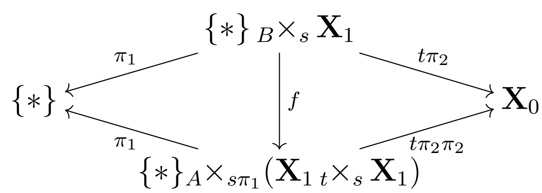
it is not surprising that we define, for all $A, B, C \in \mathbf{X}_0$, a composition map $\widetilde{\mu}$ to be the (Kleisli-inspired) composite

$$\begin{array}{c} \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}) \\ \Big\downarrow {\scriptstyle \mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(T,T)\times \mathrm{id}} \\ \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A}) \times \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{B}) \\ \Big\downarrow {\scriptstyle \circ^{\mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)}_{TT\vec{A},T\vec{B},\vec{C}}} \\ \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, TT\vec{A}) \\ \Big\downarrow {\scriptstyle \mathrm{Span}(\mathbf{Set})(\{*\},\mathbf{X}_0)(\vec{C},\mu)} \\ \mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{C}, T\vec{A}) \end{array}$$

Describing these concretely first requires an interpretation of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T\vec{B}, TT\vec{A}).$$

Its elements are span morphisms of the form



and are therefore assignments of arrows f with source B to composable pairs of arrows with source A and target tf :

$$(B \to C) \longmapsto (A \to C' \to C).$$

The morphism $\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(T, T)$, then, is defined by

$$\Big[f : A \to B\Big] \longmapsto \Big[(f, -) : (g : B \to C) \longmapsto (f : A \to B,\ g : B \to C)\Big]$$

We can then compute this composite

$$(f : A \to B, g : B \to C) \mapsto ((f, -), g) \mapsto (f, g)(g) \mapsto g \circ f = \mu(f, g);$$

that is, the composition defined by μ coincides with the T-Kleisli composition within the hom-categories.

Finally, with the above structures guiding our intuition, we define a restriction monad in certain bicategories.

Definition 3.1.1. Suppose that $\mathcal{B}$ is a bicategory with an involution on 1-cells containing an object E satisfying $\mathcal{B}(E,E)_0 \cong \mathcal{B}_0$. We call a 1-cell A E-*elemental* whenever $A^*A \cong \mathrm{id}_E$.

A restriction monad on x in $\mathcal{B}$ is a monad (T, η, μ) on x together with a family

$$\rho_{A,B} : \mathcal{B}(E,x)(B,TA) \to \mathcal{B}(E,x)(A,TA)$$

of functions indexed by E-elemental one-cells $A, B : E \to x$. Let Δ be the diagonal map $\Delta_X : X \to X \times X$, $\tau_{X,Y} : X \times Y \cong Y \times X$ (in **Set**), and define

$$\widetilde{\mu}_{A,B,C} : \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(C,TB) \to \mathcal{B}(E,x)(C,TA)$$

by the composite

$$\Big(\mathcal{B}(E,x)(T,T) \times \mathrm{id}\Big).\Big(\circ^{\mathcal{B}(E,x)}_{TTA,TB,C}\Big).\mathcal{B}(E,x)(C,\mu_A).$$

For every triple of 1-cells $A, B, C : E \to x$, we require that the following diagrams commute:

(R.1)

$$\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA) & \xrightarrow{\quad\Delta\quad} & \mathcal{B}(E,x)(B,TA) \times \mathcal{B}(E,x)(B,TA) \\
\Big\| & & \Big\downarrow{\scriptstyle \rho\times\mathrm{id}} \\
\mathcal{B}(E,x)(B,TA) & \xleftarrow[\quad\widetilde{\mu}\quad]{} & \mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,x)(B,TA)
\end{array}$$

(R.2)

$$
\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & \xrightarrow{\rho\times\rho} & \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(A,TA) \\
\downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\widetilde{\mu}} \\
\mathcal{B}(E,x)(C,TA)\times\mathcal{B}(E,x)(B,TA) & & \mathcal{B}(E,x)(A,TA) \\
\downarrow{\scriptstyle\rho\times\rho} & \nearrow{\scriptstyle\widetilde{\mu}} & \\
\mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(A,TA) & &
\end{array}
$$

(R.3)

$$
\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & \xrightarrow{\rho\times\mathrm{id}} & \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(C,TA) \\
 & & \downarrow{\scriptstyle\widetilde{\mu}} \\
\downarrow{\scriptstyle\rho\times\rho} & & \mathcal{B}(E,x)(C,TA) \\
 & & \downarrow{\scriptstyle\rho} \\
\mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}]{} & \mathcal{B}(E,x)(A,TA)
\end{array}
$$

(R.4)

$$
\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & \xrightarrow{\mathrm{id}\times\rho} & \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TB) \\
\downarrow{\scriptstyle \Delta\times\mathrm{id}} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & & \downarrow{\scriptstyle \widetilde{\mu}} \\
\downarrow{\scriptstyle \mathrm{id}\times\widetilde{\mu}} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & & \\
\downarrow{\scriptstyle \mathrm{id}\times\rho} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}.\tau]{} & \mathcal{B}(E,x)(B,TA)
\end{array}
$$

$\diamondsuit$

Proposition 3.1.2. *Small restriction categories are in one-to-one correspondence with restriction monads in* Span(**Set**).

Proof. Consider a monad T on $\mathbf{X}_0$ in Span(**Set**) as above, with its corresponding category denoted by **X**. An set E with Span(**Set**)$(E,E)\cong\mathbf{Set}_0$ is a single element set and therfore is (bijective to) the intial object in **Set**. E therefore chooses elements of sets and an E-element 1-cell is neccesarily of the form $\vec{A}$ for some set A. We have seen that

(1) the ρ maps assign each arrow to an endomorphism on its source,

(2) the hom-sets Span(**Set**)$(\{*\},\mathbf{X}_0)(\vec{B},T\vec{A})$ can be identified with $\mathbf{X}(A,B)$, and

(3) the T-Kleisli composition coincides with μ (that is, composition in **X**).

We may then verify that the axioms (R.1) through (R.4) are satisfied, making ρ a restriction structure on X.

(R.1):

$$f = \widetilde{\mu}.(\rho \times \mathrm{id})(f, f) = \widetilde{\mu}.(\rho(f), f) = f\,\rho(f)$$

(R.2):

$$\widetilde{\mu}.(\rho \times \rho).\tau(f, g) = \widetilde{\mu}.(\rho \times \rho)(g, f) = \widetilde{\mu}.(\rho \times \rho)(\rho(g), \rho(f)) = \rho(f)\,\rho(g)$$

is equal to $\rho(g)\,\rho(f)$.

(R.3):

$$\rho.\widetilde{\mu}.(\rho \times \mathrm{id})(f, g) = \rho.\widetilde{\mu}(\rho(f), g) = \rho(g\,\rho(f))$$

is equal to $\rho(g)\,\rho(f)$.

(R.4):

$$\begin{aligned} \mu.\tau.(\mathrm{id} \times \rho).(\mathrm{id} \times \widetilde{\mu}).(\Delta \times \mathrm{id})(f, g) &= \mu.\tau.(\mathrm{id} \times \rho).(\mathrm{id} \times \widetilde{\mu})(f, f, g) \\ &= \mu.\tau.(\mathrm{id} \times \rho)(f, gf) \\ &= \mu.\tau(f, \rho(gf)) \\ &= f\,\rho(gf) \end{aligned}$$

is equal to

$$\mu.(\mathrm{id} \times \rho)(f, g) = \mu(f, \rho(g)) = \rho(g)\, f.$$

Given a small restriction category $\mathbf{X}$, define a restriction monad by the following data:

- Its 0-cell is $\mathbf{X}_0$.
- We define $T : \mathbf{X}_0 \xleftarrow{s} \mathbf{X}_1 \xrightarrow{t} \mathbf{X}_0$.
- We define $\rho_{A,B} : f \mapsto \overline{f}$.

In this direction, it is immediate that this data comprises a restriction monad and establishes a one-to-one correspondence between small restriction categories and restriction monads in Span($\mathbf{Set}$). □

Proposition 3.1.3. *Restriction monads in* $\mathbf{Set}$-Mat *are in one-to-one correspondence with small restriction categories.*

Proof. $\mathbf{Set}$-Mat (see [3]) is the bicategory whose 0-cells are sets, whose 1-cells are matrices $A \to B$ encoded by functors $B \times A \to \mathbf{Set}$, and whose 2-cells are matrix morphisms (matrices of functions defining a componentwise morphism). Composition of two 1-cells is done by "matrix multiplication"; for matrices $A \xrightarrow{M} B \xrightarrow{M'} C$, the composite $M' \otimes M : A \to C$ is defined componentwise by

$$(M' \otimes M)(c, a) = \sum_{b \in B} M'(c, b) \times M(b, a)$$

A monad $T : \mathbf{X}_0 \to \mathbf{X}_0$ in **Set**-Mat then is a matrix encoded by a functor $T : \mathbf{X}_0 \times \mathbf{X}_0 \to \mathbf{Set}$. Such monads are well known to be in one-to-one correspondence with small categories with a set of objects $\mathbf{X}_0$ and $\mathbf{X}(A, B) = T(B, A)$. The multiplication maps

$$\mu_{A,B,C} : \sum_{B \in \mathbf{X}_0} T(C, B) \times T(B, A) \to T(C, A)$$

then encodes composition of composable pairs of arrows, and the unit maps

$$\eta_A : A \mapsto 1_A \in T(A, A)$$

encode the identity morphisms.

Each object $A \in \mathbf{X}_0$ can be identified with a (row) matrix $\vec{A} : \{*\} \to \mathbf{X}_0$ which is empty everywhere except at the "Ath" component and $\{*\}$ at the "Ath" component. We note that, similarly to the span case, E will be any one-element set and all E-elemental 1-cells will be of this form. Then the composite $T\vec{A}$ is a row matrix containing of $T(A, B)$, one for each $B \in \mathbf{X}_0$. A morphism then from $\vec{B}$ to $T\vec{A}$ will be a row vector which is empty except at $T(B, A)$, which can be identified with the set $T(B, A)$ (and thus $\mathbf{X}(A, B)$) itself.

The family of restriction maps

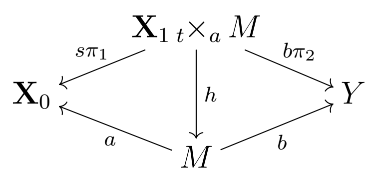
$$\rho_{A,B} : \mathbf{Set}\text{-Mat}(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \to \mathbf{Set}\text{-Mat}(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

then corresponds as above to an assignment of arrows $f : A \to B$ to endomorphisms $\overline{f} := \rho(f) : A \to A$ satisfying the restriction axioms. □

Algebras for Restriction Monads

In this section, we define an algebra for a restriction monad. Much like the monads, we must be able to keep track of "sources" so that we can define "restrictions" on those sources. We must also have ways of taking diagonals of "elements" of our algebras and generalized structure maps to move between these structures.

Suppose that T is an ordinary monad in Span(**Set**) and that $\mathbf{X}$ is its corresponding small category. Recall that algebras (S, h) for T are in correspondence with a class of right-$\mathbf{X}$ modules on the apex set of $S = \mathbf{X}_0 \xleftarrow{a} M \xrightarrow{b} Y$ with the action given by $h : ST \Rightarrow S$



by defining $\alpha \cdot f = h(f, \alpha)$ for all $(f, \alpha) \in \mathbf{X}_1\,{}_t\!\times_a M$.

Similarly to how we identify the hom-set $\text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ with $\mathbf{X}(A, B)$, we identify $\text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with the module set $S(B, A)$. This identification allows us to define a restriction operator r as a family of set functions

$$r_{A,B} : \text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}) \to \text{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

of module elements $\alpha : A \longrightarrow\!\!\!\!\!\!\!\!\mapsto B$ to arrows $r(\alpha) : A \to A$. We will require that each $r(\alpha)$ is a restriction idempotent of $\mathbf{X}$ and we will therefore require that $\text{Im}(r_{A,B}) \subseteq \cup_{A':1\to x}\text{Im}(\rho_{A,A'})$.

Identifying $\text{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$ with $S(B, A)$, we must therefore consider how to "h-act" with elements of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A}') \times \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

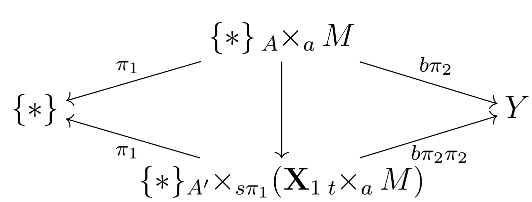
We define, for all $A, B, C \in \mathbf{X}_0$, an h-action map $\widetilde{h}$ to be the composite

$$\begin{array}{c}
\mathrm{Span}(\mathbf{Set})(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A}') \times \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}) \\
\Big\downarrow {\scriptstyle \mathrm{Span}(\mathbf{Set})(\{*\}, S\mathbf{X}_0)(S,S)\times \mathrm{id}} \\
\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(S\vec{A}, ST\vec{A}') \times \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}) \\
\Big\downarrow {\scriptstyle \circ^{\mathrm{Span}(\mathbf{Set})(\{*\},Y)}_{ST\vec{A}', S\vec{A}, \vec{B}}} \\
\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, ST\vec{A}') \\
\Big\downarrow {\scriptstyle \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, h)} \\
\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A}')
\end{array}$$

Describing these concretely first requires an interpretation of the set

$$\mathrm{Span}(\mathbf{Set})(\{*\}, Y)(S\vec{A}, ST\vec{A}').$$

Its elements are span morphisms of the form



and are therefore assignments of elements $\alpha \in M$ with source A to actionable pairs with source A' and target $b\alpha$:

$$(A \longrightarrow\!\!\!\!\!\!\mapsto B) \longmapsto (A' \to A \longrightarrow\!\!\!\!\!\!\mapsto B).$$

The morphism Span(**Set**)($\{*\}, S\mathbf{X}_0)(S,S)$, then, is defined by

$$\Big[f : A' \to A\Big] \longmapsto \Big[(f,-) : (\alpha : A \longrightarrow\!\!\!\!\!\!\mapsto B) \longmapsto (f,\alpha)\Big]$$

We can then compute this composite

$$(f : A' \to A, \alpha : A \longrightarrow\!\!\!\!\!\!\mapsto B) \mapsto ((f,-),\alpha)) \mapsto (f,\alpha) \mapsto \alpha \cdot f = h(f,\alpha);$$

that is, the action defined by h coincides with the $\widetilde{h}$ action within the hom-categories.

Definition 3.1.4. Suppose that $\mathcal{B}$ is a bicategory with an involution on 1-cells containing an object E satisfying $\mathcal{B}(E,E)_0 \cong \mathcal{B}_0$.

An algebra for a restriction monad (T,η,μ,ρ) on x in $\mathcal{B}$ is an algebra (S,h) for (T,η,μ) together with a family $r_{A,B} : \mathcal{B}(E,y)(B,SA) \to \mathcal{B}(E,x)(A,TA)$ of functions, indexed by 1-cells of the form $A : 1 \to x$ and $B : 1 \to y$, with $\mathrm{Im}(r_{A,B}) \subseteq \cup_{A':1\to x}\mathrm{Im}(\rho_{A,A'})$. Define

$$\widetilde{h}_{A',A,B} : \mathcal{B}(E,x)(A,TA') \times \mathcal{B}(E,y)(B,SA) \to \mathcal{B}(E,y)(B,SA')$$

by the composite

$$\Big(\mathcal{B}(1,Sx)(S,S) \times \mathrm{id}\Big).\Big(\circ^{\mathcal{B}(E,y)}_{STA',SA,B}\Big).\mathcal{B}(E,y)(B,h_{A'})$$

For every triple of 1-cells $A, B, C : E \to x$, we require that the following diagrams commute:

(R.1)

$$\begin{array}{ccc} \mathcal{B}(E,y)(B,SA) & \xrightarrow{\Delta} & \mathcal{B}(E,y)(B,SA)\times\mathcal{B}(E,y)(B,SA) \\ \Vert & & \downarrow{\scriptstyle r\times \mathrm{id}} \\ \mathcal{B}(E,y)(B,SA) & \xleftarrow[\widetilde{h}]{} & \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,y)(B,SA) \end{array}$$

(R.3)

$$\begin{array}{ccc} \mathcal{B}(E,y)(B,SA)\times\mathcal{B}(E,y)(B',SA) & \xrightarrow{r\times \mathrm{id}} & \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,y)(B',SA) \\ & & \downarrow{\scriptstyle \widetilde{h}} \\ {\scriptstyle r\times r}\downarrow & & \mathcal{B}(E,y)(B',SA) \\ & & \downarrow{\scriptstyle r} \\ \mathcal{B}(E,x)(A,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}]{} & \mathcal{B}(E,x)(A,TA) \end{array}$$

(R.4)

$$\begin{array}{ccc} \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & \xrightarrow{\mathrm{id}\times r} & \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TB) \\ {\scriptstyle \Delta\times\mathrm{id}}\downarrow & & \\ \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & & \\ {\scriptstyle \mathrm{id}\times\widetilde{h}}\downarrow & & \downarrow{\scriptstyle \widetilde{\mu}} \\ \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & & \\ {\scriptstyle \mathrm{id}\times r}\downarrow & & \\ \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}.\tau]{} & \mathcal{B}(E,x)(B,TA) \end{array}$$

$\diamondsuit$

The restriction algebra constructed for a restriction monad T in Span(**Set**) corresponding to a small restriction category $\mathbf{X}$ can therefore be considered as a right-$\mathbf{X}$ module on a set together with some restriction structure (with respect to the action). We call such a structure a *restriction module*.

3.2 Restriction Modules

Definition 3.2.1. A *restriction (left* $\mathbf{Y}$*-, right* $\mathbf{X}$*-bi)module* $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\mid\;\; \mathbf{Y}$ is a collection

$$\{\varphi(y,x) : y \in \mathbf{Y}_0, x \in \mathbf{X}_0\}$$

of sets indexed by the objects of $\mathbf{X}$ and $\mathbf{Y}$ together with:

- for all objects $y, y' \in \mathbf{Y}$ and $x, x' \in \mathbf{X}$, a pair of action maps

$$\lambda^{\varphi}_{y',y,x} : \mathbf{Y}(y,y') \times \varphi(y,x) \longrightarrow \varphi(y',x)$$
$$\rho^{\varphi}_{y,x,x'} : \varphi(y,x) \times \mathbf{X}(x',x) \longrightarrow \varphi(y,x')$$

  (subscripts will be omitted when no ambiguity exists) such that the following diagrams commute:

(Mod.1) (Associativity Laws)

$$\begin{array}{ccc}
\mathbf{Y}(y,y') \times \mathbf{Y}(y',y'') \times \varphi(y,x) & \xrightarrow{1\times\lambda} & \mathbf{Y}(y',y'') \times \varphi(y',x) \\
\downarrow{\scriptstyle \circ\times 1} & & \downarrow{\scriptstyle \lambda} \\
\mathbf{Y}(y,y'') \times \varphi(y,x) & \xrightarrow[\lambda]{} & \varphi(y'',x)
\end{array}$$

$$\begin{array}{ccc}
\varphi(y,x)\times\mathbf{X}(x'',x)\times\mathbf{X}(x,x') & \xrightarrow{\rho\times 1} & \varphi(y,x'')\times\mathbf{X}(x'',x') \\
{\scriptstyle 1\times\circ}\downarrow & & \downarrow{\scriptstyle \rho} \\
\varphi(y,x)\times\mathbf{X}(x'',x) & \xrightarrow[\rho]{} & \varphi(y,x'')
\end{array}$$

$$\begin{array}{ccc}
\mathbf{Y}(y,y')\times\varphi(y,x)\times\mathbf{X}(x',x) & \xrightarrow{\lambda\times 1} & \varphi(y',x)\times\mathbf{X}(x',x) \\
{\scriptstyle 1\times\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\
\mathbf{Y}(y,y')\times\varphi(y,x') & \xrightarrow[\lambda]{} & \varphi(y',x')
\end{array}$$

(Mod.2) (Unit Laws)

$$\begin{array}{ccc}
\mathbf{Y}_0\times\varphi(y,x) & \xrightarrow{u\times 1} & \mathbf{Y}(y,y)\times\varphi(y,x) \\
& {\scriptstyle y\times -}\nwarrow & \downarrow{\scriptstyle \lambda} \\
& & \varphi(y,x)
\end{array}
\qquad
\begin{array}{ccc}
\varphi(y,x)\times\mathbf{X}(x,x) & \xleftarrow{1\times u} & \varphi(y,x)\times\mathbf{X}_0 \\
{\scriptstyle \rho}\downarrow & \nearrow{\scriptstyle -\times x} & \\
\varphi(y,x) & &
\end{array}$$

We will write both $\lambda(g,\alpha)$ and $\rho(\alpha,f)$ using the dot notation $g\cdot\alpha$ and $\alpha\cdot f$.

- a map assigning each $\alpha\in\varphi(y,x)$ to some $\overline{\alpha}: x\to x$ in $\mathbf{X}$ satisfying:

(RMod.0) for each $\alpha\in\varphi(y,x)$, $\overline{\alpha}$ is a restriction idempotent of $\mathbf{X}$.

(RMod.1) for each $\alpha\in\varphi(y,x)$, $\alpha\cdot\overline{\alpha}=\alpha$;

(RMod.3) for each $\alpha\in\varphi(y,x)$ and $\beta\in\varphi(y',x)$, $\overline{\alpha\cdot\overline{\beta}}=\overline{\alpha}\circ\overline{\beta}$;

(RMod.4) (a) for each $\alpha\in\varphi(y,x)$ and $f: x'\to x$ in $\mathbf{X}$, $\overline{\alpha}\circ f=f\circ\overline{\alpha\cdot f}$;

(b) for each $\alpha\in\varphi(y,x)$ and $g: y\to y'$ in $\mathbf{Y}$, $\overline{g}\cdot\alpha=\alpha\cdot\overline{g\cdot\alpha}$. $\diamondsuit$

Note the omission of (RMod.2); since (RMod.0) requires that the restrictions must be restriction idempotents in the source category, the idempotents must already commute.

Proposition 3.2.2. *Let* $\mathbf{X}$ *be a small restriction category and let* (T, η, μ, ρ) *denote its corresponding restriction monad in* $\mathrm{Span}(\mathbf{Set})$. *An algebra*

$$(S = \mathbf{X}_0 \xleftarrow{a} M \xrightarrow{b} Y, h, r)$$

corresponds to a right-$\mathbf{X}$ *restriction module, whose right* $\mathbf{X}$*-action is defined by h-evaluation.* Y *is a set viewed as a discrete category.*

Proof. We will define a right $\mathbf{X}$-restriction module

$$\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\!\mapsto Y$$

by

$$\varphi(B, A) = \{\alpha \in M \,|\, a\alpha = A, b\alpha = B\} = \mathrm{Span}(\mathbf{Set})(\{*\}, Y)(\vec{B}, S\vec{A})$$

That the h-maps are span morphisms will force any well defined action to be contained in some $\varphi(y, x)$. We use the dot notation to denote the evaluation of the h-maps, which will be the right action of $\mathbf{X}$ on M. We define the restriction of each element $\alpha \in M$ by $\overline{\alpha} = r_{A,B}(\alpha) \in \mathrm{Im}(\rho_{A,B})$. (RMod.0) is then satisfied by definition, since $\mathrm{Im}(\rho)$ contains exactly the restriction idempotents of $\mathbf{X}$. We prove now that the remaining restriction axioms are satisfied.

(RMod.1) The commutativity of

$$\begin{array}{ccc}
\mathcal{B}(E,y)(B,SA) & \xrightarrow{\Delta} & \mathcal{B}(E,y)(B,SA) \times \mathcal{B}(E,y)(B,SA) \\
\| & & \downarrow{\scriptstyle r\times\mathrm{id}} \\
\mathcal{B}(E,y)(B,SA) & \xleftarrow[\widetilde{h}]{} & \mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,y)(B,SA)
\end{array}$$

implies that, for all $\alpha \in M$,

$$\alpha = \widetilde{h}.(r \times \mathrm{id}).\Delta(\alpha) = \widetilde{h}.(r \times \mathrm{id}).(\alpha, \alpha) = \widetilde{h}.(r(\alpha), \alpha) = \alpha \cdot \overline{\alpha}$$

(RMod.3) The commutativity of

$$\begin{array}{ccc}
\mathcal{B}(E,y)(B,SA) \times \mathcal{B}(E,y)(B',SA) & \xrightarrow{r\times\mathrm{id}} & \mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,y)(B',SA) \\
 & & \downarrow{\scriptstyle \widetilde{h}} \\
{\scriptstyle r\times r}\downarrow & & \mathcal{B}(E,y)(B',SA) \\
 & & \downarrow{\scriptstyle r} \\
\mathcal{B}(E,x)(A,TA) \times \mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}]{} & \mathcal{B}(E,x)(A,TA)
\end{array}$$

implies that, for all $\alpha, \beta \in B$ with $a\alpha = a\beta$, $\widetilde{\mu}.(r \times r)(\alpha, \beta) = \overline{\alpha} \circ \overline{\beta}$ is equal to

$$r.\widetilde{h}.(r \times \mathrm{id})(\alpha, \beta) = r.\widetilde{h}(r(\alpha), \beta) = r(\beta \cdot r(\alpha)) = \overline{\beta \cdot \overline{\alpha}}$$

(RMod.4) The commutativity of

$$\begin{array}{ccc}
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & \xrightarrow{\mathrm{id}\times r} & \mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TB) \\
\downarrow{\scriptstyle \Delta\times\mathrm{id}} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TB) & & \\
\downarrow{\scriptstyle \mathrm{id}\times\widetilde{h}} & & \downarrow{\scriptstyle \widetilde{\mu}} \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(C,TA) & & \\
\downarrow{\scriptstyle \mathrm{id}\times r} & & \\
\mathcal{B}(E,x)(B,TA)\times\mathcal{B}(E,x)(A,TA) & \xrightarrow[\widetilde{\mu}.\tau]{} & \mathcal{B}(E,x)(B,TA)
\end{array}$$

implies that, for all $f \in \mathbf{X}_1$ and $\alpha \in M$ with $a\alpha = tf$,

$$\widetilde{\mu}.(\mathrm{id}\times r)(f,\alpha) = \widetilde{\mu}(f,r(\alpha)) = \overline{\alpha}\circ f$$

$$\begin{aligned}
\widetilde{\mu}.\tau.(\mathrm{id}\times r).(\mathrm{id}\times\widetilde{h}).(\Delta\times\mathrm{id})(f,\alpha) &= \widetilde{\mu}.\tau.(\mathrm{id}\times r).(\mathrm{id}\times\widetilde{h})(f,f,\alpha) \\
&= \widetilde{\mu}.\tau.(\mathrm{id}\times r)(f,\alpha\cdot f) \\
&= \widetilde{\mu}.\tau(f,r(\alpha\cdot f)) \\
&= f\circ\overline{\alpha\cdot f}
\end{aligned}$$ □

We end this section with a short discussion on restriction presheaves, as introduced by Lin in his thesis and detailed in a recent preprint [17].

Definition 3.2.3. Let $\mathbf{X}$ be a restriction category. A *restriction presheaf* on $\mathbf{X}$ is a contravariant functor $P : \mathbf{X}^{\mathrm{op}} \to \mathbf{Set}$ together with a family of restriction assignments

$$P(x) \to \mathbf{X}(x,x) : f \mapsto \overline{f}$$

satisfying

(A0) $\overline{f}$ is a restriction idempotent of $\mathbf{X}$;

(A1) $P(\overline{f})(f) = f$;

(A2) $\overline{P(\overline{f})(g)} = \overline{g} \circ \overline{f}$, where $\overline{f} : x \to x$ is a restriction idempotent in $\mathbf{X}$;

(A3) $\overline{g} \circ f = f \circ \overline{P(f)(g)}$, for each $f : y \to x$ in $\mathbf{X}$. ♢

Proposition 3.2.4. *Given one of Lin's restriction presheaves P on $\mathbf{X}$, one can construct a right-$\mathbf{X}$ restriction module.*

Proof. Define $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\mapsto \{*\}$ by $\varphi(y, x) = P(x)$.

The action is defined, as usual, by $x \cdot f = P(f)(x)$ and the assignment of each $\alpha \in \varphi(*, x)$ to some $\overline{\alpha} : x \to x$ in $\overline{\mathbf{X}}$ is given by Lin's family $P(x) \to \mathbf{X}(x, x)$ of restriction assignments, which will then satisfy the conditions making φ a restriction module. □

3.3 Future Work

The constructions of restriction monads from small restriction categories and vice versa detailed in this chapter are defined only on the objects of (what we would suppose to be) the categories of interest. These constructions are unfortunately not functorial (not even in the ordinary case, without a restriction structure), since the maps between monads do not give functors between their corresponding categories (unless they each have the same set of objects). To make these constructions functorial, one needs to keep track of how the source and target 0-cells are interacting.

Indeed, this construction *is* functorial when defined on the double category of spans [15]. While the conversion of *this* bicategorical definition into double monads and double categories is straightforward, future work includes attempting to take advantage of this additional information and define the *right* double categorical definition in the hope that the 2-cell ρ can be defined a little more cleanly and to establish an adjunction (or similar) in the appropriate setting.

In the longer term, I would like to sort out the algebraic structure of restriction algebras for arbitrary restriction monads. This is certainly open-ended and exploratory, but following are three short-term goals for applying this framework.

Question. A restriction monad in Span(**Set**) is a small restriction category. This leads immediately to questioning how similar structures are related: Is an ordinary monad in **rCat** a restriction monad? Is an ordinary monad in Span(**rCat**) a restriction category?

Question. The work of Cockett and Cruttwell [8] introduces the data of a restriction tangent bundle, which gives a categorical model of differentiation in a suitable abstract setting. A tangent bundle is a restriction endofunctor equipped with some structure reminiscent of a structured comonad (in the appropriate setting). I would like to know how much of the restriction structure can be moved into the comonad.

Question. Partially invertible computation has been modelled recently by Giles [18] using Cartesian inverse categories. I would like to better understand the role that the Cartesian structure plays in computation, and how this model compares to a restriction monad on an inverse category.

Chapter 4: Geometry and Topology

This chapter explores the equipment of certain restriction categories with topological structure. We show that any inverse category is equivalent to what we call a top-heavy locally inductive groupoid and this groupoid can be naturally equipped with what Lawson calls an Ehresmann topology.

4.1 Ehresmann-Schein-Nambooripad Theorem

The Ehresmann-Schein-Nambooripad (ESN) Theorem asserts the existence of an equivalence between the category of inverse semigroups (with semigroup homomorphisms) and the category of (inductive) ordered groupoids (with inductive functors). A groupoid is called ordered in this context if there is a compatible (functorial) order on both objects and arrows with a notion of restriction on the arrows such that an arrow $f\colon A \to B$ has a unique restriction $f'\colon A' \to B'$ with $f' \leq f$ for any object $A' \leq A$. For the precise definition see Definition 4.1.1, but a cateogry theorist may like to think of these as groupoids internal to the category of posets with some additional properties. Ordered functors between these are functors that preserve the order. Furthermore, an ordered groupoid is called inductive when the objects form a meet-semilattice and an ordered functor is inductive when it preserves the meets. The correspondence of the ESN Theorem is directly extendable to inverse semigroups and prehomomorphisms when one takes ordered functors, rather than inductive functors, between the inductive groupoids.

This theorem has been extended to various larger classes of semigroups such as regular semigroups [35, 36, 37], two-sided restriction semigroups (also called Ehresmann semigroups) [29] and more general restriction groups [20] with either semigroup homomorphisms or $\wedge$- or $\vee$-prehomomorphisms. The main ideas in this context have focused on either changing the requirement for a meet-semilattice structure to a different order structure on the objects of the groupoid, or on generalizing to inductive categories rather than groupoids.

Our approach here is to generalize this equivalence in a different direction. Semigroups can be viewed as single-object semicategories and we want to obtain a 'multi-object' version of the correspondence. As groupoids can be thought of as the multi-object version of groups, we think of inverse categories as a multi-object version of inverse semigroups. In this thesis, we prove a new generalization of the ESN theorem which extends the result to inverse categories. Since we are generalizing the concept of *inverse* semigroup, we will remain within the category of groupoids. They will still be ordered, but the order structure will only be locally inductive in a suitable sense: the objects need to form a disjoint union of meet-semilattices. Since inverse categories have identities, we further require that the meet-semilattices have a top-element. If we instead generalize to inverse semicategories, this last requirement is not needed. Locally inductive functors, ordered functors that preserve all meets that exist, will correspond to functors of inverse semicategories (Corollary 4.2.16). We will also show that the category of inverse categories and oplax functors is equivalent to the category of top-heavy locally inductive groupoids and ordered functors, generalizing the classical result that the category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors (Theorem 4.2.19).

The groupoid we construct for an inverse category was independently considered in the work of Linckelmann [30] on category algebras. Linckelmann observes that this groupoid has the same category algebra as the original inverse category, giving the category algebra of an inverse category the structure of a groupoid algebra: a groupoid algebra over a commutative ring is a direct product of matrix rings. In this thesis, we introduce this groupoid with an ordered structure and observe the important characterizing properties of the order structure to obtain an equivalence of categories between the category of inverse categories and the category of these top-heavy locally inductive groupoids.

From the semigroup perspective, this raises the question of whether there are appropriate multi-object versions of the other classes of semigroups mentioned above which then may be shown to be equivalent to appropriate versions of locally inductive categories.

Ordered Groupoids

Inductive groupoids are a class of groupoids whose arrows are equipped with a partial order satisfying certain conditions and whose objects form a meet-semilattice. Charles Ehresmann used ordered groupoids to model pseudogroups while inverse semigroups (introduced by Gordon Preston [38]) were concurrently used as an alternate model for pseudogroups. Ehresmann was certainly aware of the connection between ordered (inductive) groupoids and inverse semigroups, as it was Ehresmann who first introduced the tensor product required to make the correspondence work. Boris Schein [39] made this connection explicit, requiring that the set of objects form a meet-semilattice, thus guaranteeing the existence of this tensor product for all arrows of the groupoid. K.S.S. Nambooripad [35, 36, 37] independently developed the theory of so-called regular systems and their correspondence to so-called regular groupoids. This theory is, in fact, more general and specializes to the correspondence between inverse semigroups and inductive groupoids. A more detailed history of inverse semigroups, inductive groupoids and their applications can be found in Hollings' [19]. In this section, we present the modern exposition of this correspondence, which can be found in Mark Lawson's book [26], where these constructions and their equivalence were first explicitly given.

Definition 4.1.1. A groupoid $\mathbf{G}$ is said to be an *ordered groupoid* whenever there is a partial order $\leq$ on its arrows satisfying the following four conditions:

(i) For each arrow $f, g \in G$, $f \leq g$ implies $f^{-1} \leq g^{-1}$.

(ii) For all arrows $a, A, b, B \in G$ such that $a \leq A$, $b \leq B$ and the composites ab and AB exist, $ab \leq AB$.

(iii) For each arrow $f : A' \to B$ in G and each object $A \leq A'$ in G, there exists a unique *restriction of f to A*, denoted $[f|_* A]$, such that $\mathrm{dom}[f|_* A] = A$ and $[f|_* A] \leq f$.

(iv) For each arrow $f : A \to B'$ in G and objects $B \leq B'$ in G, there exists a unique *corestriction of f to B*, denoted $[B\,_*|f]$, such that $\mathrm{cod}[B\,_*|f] = B$ and $[B\,_*|f] \leq f$.

An ordered groupoid is said to be an *inductive groupoid* whenever its objects form a meet-semilattice. ♢

Though it is sometimes convenient to explicitly give both the restrictions and corestrictions in an ordered groupoid, the following proposition makes it necessary only to include one of them in any proofs.

Proposition 4.1.2 ([26]). *In Definition 4.1.1, conditions (iii) and (iv) are equivalent.*

□

Definition 4.1.3. Let $\mathbf{G}$ be an ordered groupoid with arrows $\alpha, \beta \in \mathbf{G}$. If $\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)$ exists, the *tensor product* $\alpha \otimes \beta$ of α and β is defined as

$$\alpha \otimes \beta = [\alpha\,|_*\,\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)][\mathrm{dom}(\alpha) \wedge \mathrm{cod}(\beta)\,{}_*|\,\beta]. \qquad \diamondsuit$$

Proposition 4.1.4 ([26]). *Let* $\mathbf{G}$ *be an inductive groupoid. This tensor product is associative and admits pseudoinverses given by the inverses in the inductive groupoid, making* $(\mathbf{G}_1, \otimes)$ *an inverse semigroup.*

Proof sketch. For any pair of arrows in $\mathbf{G}$, one can show that the set

$$\langle \alpha, \beta \rangle = \{(\alpha', \beta') |\, \mathrm{cod}(\alpha') = \mathrm{dom}(\beta'), \alpha' \leq \alpha, \beta' \leq \beta\}$$

contains a unique maximal element (α', β') with $\alpha \otimes \beta = \beta' \circ \alpha'$. Since defined by composition, this tensor product is therefore associative. □

Proposition 4.1.5. *For all objects* $A \leq B$ *of an ordered groupoid,* $[1_B\,|_*\,A] = 1_A = [A\,{}_*|\,1_B]$.

Proof. The partial order on arrows induces the partial order on the objects of an ordered groupoid. Since an object of a category is identified by the identity arrow on that object, we have that $1_A \leq 1_B$. Since the (co)domain of 1_A is A, we have $[1_B \,|_* \, A] = 1_A = [A \,_*| \, 1_B]$ by the uniqueness of (co)restrictions □

Definition 4.1.6. A morphism $F : \mathbf{G} \to \mathbf{H}$ of ordered groupoids (an *ordered functor*) is a functor such that, for all arrows $f \leq g$ in $\mathbf{G}$, $F(f) \leq F(g)$ in $\mathbf{H}$. An ordered functor between inductive groupoids is said to be *inductive* whenever it preserves the meet structure on objects. ♢

Notation. We denote the category of ordered groupoids and ordered functors by **oGrpd** and the category of inductive groupoids and inductive functors by **iGrpd**.

We will now briefly review Lawson's description of functorial constructions that form the equivalence of categories between the category of inverse semigroups and the category of inductive groupoids. We remind the reader that full details can be found in [26].

Construction 4.1.7 (Inverse Semigroups to Inductive Groupoids)**.** Given an inverse semigroup $(S, \bullet)$, define an inductive groupoid $\mathcal{G}(S)$ with the following data:

- Objects: $\mathcal{G}(S)_0 = E(S)$, the idempotents in S. Since S is an inverse semigroup, $E(S)$ is a meet-semilattice with meets given by the product in S.

- Arrows: For each element $s \in S$, there is an arrow $s : s^\bullet s \to ss^\bullet$ (we remind the reader that $s^\bullet$ denotes the partial inverse of s). Composition is given by multiplication in S and identities are the elements of $E(S)$.

- Inverses: For each arrow $s : s^\bullet s \to ss^\bullet$ in $\mathcal{G}(S)$, define $s^{-1} = s^\bullet$, its pseudoinverse in S.

- The partial order on arrows is given by the natural partial order ($s \leq t$ if and only if $s = te$ for some idempotent e) on the elements of S. It can be checked that this partial order satisfies conditions (i) and (ii) of an ordered groupoid.

- The (co)restrictions are also given by multiplication in S. This can be checked to satisfy condition (iii) of an ordered groupoid. $\diamondsuit$

Construction 4.1.8 (Inductive Groupoids to Inverse Semigroups)**.** Given an inductive groupoid $(\mathbf{G}, \leq)$, define an inverse semigroup $\mathcal{S}(G)$ whose elements are the arrows of $\mathbf{G}$ and whose multiplication is given by the tensor product. This is an inverse semigroup operation with inverses those from $\mathbf{G}$ (Proposition 4.1.4). $\diamondsuit$

Theorem 4.1.9 (ESN, [26])**.** *The constructions $\mathcal{G}$ and $\mathcal{S}$ are functorial and form an equivalence of categories*

$$\mathbf{iGrpd} \underset{\mathcal{G}}{\overset{\mathcal{S}}{\rightleftarrows}} \mathbf{iSgp}$$

$\square$

4.2 Ehresmann-Schein-Nambooripad Theorem for Inverse Categories

In this section, we introduce the notion of top-heavy locally inductive groupoids: ordered groupoids whose objects may be partitioned into meet-semilattices, each of which contain a top element. We will then give functorial constructions of top-heavy locally inductive groupoids from inverse categories, and vice versa. These constructions will then be seen to give an equivalence of categories between **iCat** and **tliGrpd** (the category of top-heavy locally inductive groupoids). The identities of an inverse category are seen to correspond to the tops of the meet-semilattices in a top-heavy locally inductive groupoid and the equivalence can thus be immediately generalized to an equivalence between the category of inverse semicategories and semifunctors and the category of locally inductive groupoids and locally inductive functors. Finally, we end this section with a short discussion of a categorical analogue of the classical result in semigroup theory that the category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors. Explicitly, we show that the category of inverse categories and oplax functors is equivalent to the category of top-heavy locally inductive groupoids and ordered functors.

Definition 4.2.1. Let A be an object of a restriction category $\mathbf{X}$. Let E_A denote the set of restrictions of all endomorphisms on A. That is,

$$E_A = \left\{ \overline{f} : A \to A | f : A \to A \in \mathbf{X} \right\}. \qquad \diamondsuit$$

Notice that, for any $f : A \to B$ in $\mathbf{X}$, we have $\overline{f} : A \to A \in E_A$, since $\overline{\overline{f}} = \overline{f}$. The reason for specifying that the restrictions in E_A come from endomorphisms in $\mathbf{X}$, then serves no use further than simply reminding us that the equivalence we are trying to establish here is based on the observation that an inverse category is, at each object, an inverse semigroup (with identity).

Proposition 4.2.2. *For each object A of a restriction category $\mathbf{X}$, E_A is a meet-semilattice with meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$. In addition, E_A has top element 1_A.*

Proof. First of all, E_A is a poset with the natural partial order inherited from $\mathbf{X}$. We now show that E_A has finite meets given by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$:

- First, it is a lower bound:

$$\overline{a}\,\overline{\overline{a}\wedge\overline{b}} = \overline{a}\,\overline{\overline{a}\,\overline{b}} = \overline{a}\,\overline{a}\,\overline{b} = \overline{a}\,\overline{b} = \overline{a}\wedge\overline{b}$$

  and thus $\overline{a}\wedge\overline{b} \leq \overline{a}$. Similarly, $\overline{a}\wedge\overline{b} \leq \overline{b}$.

- This lower bound is unique up to isomorphism (equality): suppose that $\overline{d}$ is such that $\overline{d} \leq \overline{a}$, $\overline{d} \leq \overline{b}$ and $\overline{a}\wedge\overline{b} \leq \overline{d}$. Then

$$\overline{d} = \overline{a}\,\overline{d} = \overline{a}\,\overline{b}\,\overline{d} = \overline{d}\,\overline{\overline{a}\,\overline{b}} = \overline{d}\,\overline{\overline{a}\wedge\overline{b}} = \overline{a}\wedge\overline{b}.$$

Finally, since $\overline{1_A} = 1_A$, $1_A \in E_A$. Also, given any $\overline{f} : A \to A$, $1_A\overline{f} = \overline{f}$ and thus $\overline{f} \leq 1_A$ and 1_A is the top element of E_A. □

Proposition 4.2.3. *For each pair of objects A and B of a restriction category $\mathbf{X}$, if $A \neq B$, then*

$$E_A \cap E_B = \varnothing.$$

Proof. If $\overline{f} \in E_A \cap E_B$, then $A = \operatorname{dom}\left(\overline{f}\right) = B$. □

We may now give the (functorial) constructions giving an equivalence between the category of top-heavy locally inductive groupoids and inverse categories.

Construction 4.2.4. Given an inverse category $\left(\mathbf{X}, \circ, \overline{(-)}\right)$, define a groupoid $(\mathcal{G}(\mathbf{X}), \bullet, \leq)$ with the following data:

- Objects: $\mathcal{G}(\mathbf{X})_0 = \coprod_{A \in \mathbf{X_0}} E_A$.

- Arrows: Every arrow in $\mathcal{G}(\mathbf{X})$ is of the form $f : \overline{f_A} \to \overline{f_B^\circ}$ for each arrow $f : A \to B$ in $\mathbf{X}$.

  – Composition: for arrows $f : \overline{f} \to \overline{f^\circ}$ and $g : \overline{g} \to \overline{g^\circ}$ with $\overline{f^\circ} = \overline{g}$, we define their composite $g \bullet f : \overline{f} \to \overline{g^\circ}$ in $\mathcal{G}(\mathbf{X})$ to be their composite in $\mathbf{X}$. This composite is indeed an arrow, for

  $$\overline{gf} = \overline{\overline{g}f} = \overline{\overline{f^\circ}f} = \overline{f}$$

  and

  $$\overline{(gf)^\circ} = \overline{f^\circ g^\circ} = \overline{\overline{f^\circ}g^\circ} = \overline{\overline{g}g^\circ} = \overline{g^\circ}.$$

  – Identities: For any object $\overline{f} : A \to A$ in $\mathcal{G}(\mathbf{X})$, define $1_{\overline{f}} = \overline{f}$ (which is well-defined since $\overline{\overline{f}} = \overline{f}$). The identity then satisfies the appropriate axiom: for each $g : \overline{g} \to \overline{g^\circ}$ with $\overline{g} = \overline{f}$ and $\overline{g^\circ} = \overline{f^\circ}$, we have $\overline{f^\circ}g = \overline{g^\circ}g = g$ and $g\overline{f} = g\overline{g} = g$.

  – Inverses: Given an arrow $f : \overline{f} \to \overline{f^\circ}$, define $f^{-1} : \overline{f^\circ} \to \overline{f}$ to be f°, the unique restricted inverse of f from $\mathbf{X}$'s inverse structure. The composites are $ff^\circ = \overline{f^\circ} = 1_{\overline{f^\circ}}$ and $f^\circ f = \overline{f} = 1_{\overline{f}}$ as required. ♢

Definition 4.2.5. An ordered groupoid is said to be a *locally inductive groupoid* whenever there is a partition $\{M_i\}_{i\in I}$ of $\mathbf{G}_0$ into meet-semilattices M_i with the property that any two comparable objects be in the same meet-semilattice M_i. A locally inductive groupoid is said to be *top-heavy* whenever each meet-semilattice M_i admits a top-element $\top_i$. ♢

Note. The requirement that any two comparable objects of a locally inductive groupoid be in the same meet-semilattice corresponds to our intuition that if the meet $A \wedge B$ of two objects A and B exists in M_i, then A and B, both sitting above this meet, should also be elements of M_i.

Definition 4.2.6. An ordered functor between locally inductive groupoids is said to be *locally inductive* whenever it preserves all meets that exist. In particular, a locally inductive functor will preserve empty meets and thus top elements and there is no requirement to define so-called "top-heavy locally inductive functors". ♢

Notation. We denote the category of locally inductive groupoids and locally inductive functors by **liGrpd** and the category of top-heavy locally inductive groupoids and locally inductive functors by **tliGrpd**.

Proposition 4.2.7. *For each inverse category* $\mathbf{X}$, $\mathcal{G}(\mathbf{X})$ *is a top-heavy locally inductive groupoid.*

Proof. Recall that the partial order on the objects $\overline{f}$ in $\mathcal{G}(\mathbf{X})$ is that which is induced by the partial order on the arrows of $\mathbf{X}$. That is, $\overline{f} \leq \overline{g}$ if and only if $\overline{f} = \overline{\overline{g}\overline{f}} = \overline{g}\overline{f}$. We now prove that this partial order gives $\mathcal{G}(\mathbf{X})$ the structure of an ordered groupoid:

(i) Suppose that f and g are arrows in $\mathcal{G}(\mathbf{X})$ with $f \leq g$. That is, we suppose that $g\overline{f} = f$ (since these are also arrows in $\mathbf{X}$). Then

$$\begin{aligned} f^\circ = (g\overline{f})^\circ = \overline{f}^\circ g^\circ = \overline{f} g^\circ = \overline{f} g^\circ g g^\circ = g^\circ g \overline{f} g^\circ \\ = g^\circ g \overline{f}\,\overline{f} g^\circ = g^\circ g \overline{f}\,\overline{f}^\circ g^\circ = g^\circ g \overline{f} (g\overline{f})^\circ = g^\circ f f^\circ \\ = g^\circ \overline{f^\circ} \end{aligned}$$

and thus $f^{-1} = f^\circ \leq g^\circ = g^{-1}$.

(ii) This follows directly from Proposition 2.0.4.

(iii) Given an arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ with an object $\overline{e} \leq \overline{\alpha}$, we define the restriction $[\alpha|_*\overline{e}]$ of α to $\overline{e}$ to be $\alpha\overline{e}$. This is indeed an arrow whose domain is $\overline{e} : \overline{\alpha\overline{e}} = \overline{\alpha}\,\overline{e} = \overline{e}$.

Also, $\alpha\overline{\alpha\overline{e}} = \alpha\overline{\alpha}\,\overline{e} = \alpha\overline{e}$, so that $\alpha\overline{e} \leq \alpha$.

If $\beta \leq \alpha$ is any other arrow with $\mathrm{dom}(\beta) = \overline{e}$, we have $\alpha\overline{\beta} = \beta$ and $\overline{\beta} = \overline{e}$, so that $\beta = \alpha\overline{e}$ and thus $[\alpha|_*\overline{e}]$ as defined is unique.

(iv) Given an arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ with an object $\overline{e} \leq \overline{\alpha^\circ}$, we define the corestriction $[\overline{e}\,_*|\alpha]$ of α to $\overline{e}$ to be $\overline{e}\alpha$. This is indeed an arrow whose codomain is $\overline{e} : \overline{(\overline{e}\alpha)^\circ} = \overline{\alpha^\circ\overline{e}} = \overline{a^\circ}\,\overline{e} = \overline{e}$.

Also, $\alpha\overline{\overline{e}\alpha} = \alpha\overline{e\alpha} = \alpha(e\alpha)^\circ e\alpha = \alpha\alpha^\circ e^\circ e\alpha = e^\circ e\alpha\alpha^\circ\alpha = \overline{e}\alpha$, so that $\overline{e}\alpha \leq \alpha$.

If $\beta \leq \alpha$ is any other arrow with $\mathrm{cod}(\beta) = \overline{e}$, we have $\beta^\circ \leq \alpha^\circ$ (property (i) of ordered groupoids) and thus $\alpha^\circ\overline{\beta^\circ} = \beta^\circ$ and $\overline{\beta^\circ} = \overline{e}$, so that $\beta^\circ = \alpha^\circ\overline{e} = (\overline{e}\alpha)^\circ$ and thus $[\overline{e}\,_*|\alpha]$ as defined is unique.

Given the choice of objects for $\mathcal{G}(\mathbf{X})$, it follows immediately from Propositions 4.2.2 and 4.2.3 that $\mathcal{G}(\mathbf{X})$ is a top-heavy locally inductive groupoid. □

The composition in $\mathcal{G}(\mathbf{X})$ of f and g exists exactly when $\overline{f} = \overline{g^\circ}$ and is defined by the composition in $\mathbf{X}$. The tensor product in $\mathcal{G}(\mathbf{X})$ is a natural extension of this composition in the sense that it exists whenever the meet $\overline{f} \wedge \overline{g^\circ}$ exists. This lemma shows that this extension is also defined by the composition in $\mathbf{X}$.

Lemma 4.2.8. *If* $\mathbf{X}$ *is an inverse category, then in* $\mathcal{G}(\mathbf{X})$ *the tensor products (when defined) are given by composition in* $\mathbf{X}$:

$$f \otimes g = fg.$$

Proof. Recall that, for any arrow f in $\mathbf{X}$, $\mathrm{dom}(f) = \overline{f}$ and $\mathrm{cod}(f) = \overline{f^\circ}$. Then

$$\begin{aligned} f \otimes g &= [f\,|_*\, \mathrm{dom}(f) \wedge \mathrm{cod}(g)]\,[\mathrm{dom}(f) \wedge \mathrm{cod}(g)\,{}_*|\, g] \\ &= \left[f\,|_*\, \overline{f} \wedge \overline{g^\circ}\right] \left[\overline{f} \wedge \overline{g^\circ}\,{}_*|\, g\right] \\ &= \left[f\,|_*\, \overline{f}\,\overline{g^\circ}\right] \left[\overline{f}\,\overline{g^\circ}\,{}_*|\, g\right] = f\overline{f}\,\overline{g^\circ}\,\overline{f}\,\overline{g^\circ}g = f\overline{f}\,\overline{g^\circ}g = fg \end{aligned}$$

□

Proposition 4.2.9. *Locally inductive functors preserve tensors.*

Proof. This follows immediately from the definition of a locally inductive functor and the fact that any ordered functor preserves restrictions and corestrictions [26, Proposition 4.1.2(1)]. □

Proposition 4.2.10. *For each functor* $F : \mathbf{X} \to \mathbf{Y}$ *between inverse categories, there exists a locally inductive functor* $\mathcal{G}(F) : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{Y})$.

Proof. We claim that $F : \mathbf{X} \to \mathbf{Y}$ induces a locally inductive functor $\mathcal{G}(F)$ between the groupoids $\mathcal{G}(\mathbf{X})$ and $\mathcal{G}(\mathbf{Y})$. Since F is a functor of inverse categories, we have, for each $\overline{f}$ in $\mathbf{X}$, that $F\overline{f} = \overline{F(f)}$ is a restriction idempotent in $\mathbf{Y}$. We can then define, for any object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, $\mathcal{G}(F)(\overline{f}) = F\overline{f}$ and this is a well-defined object function.

Given an arrow $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$, we define

$$\mathcal{G}(F)(f) := \left[F(f) : F\left(\overline{f}\right) \to F\left(\overline{f^\circ}\right)\right] = \left[F(f) : \overline{F(f)} \to \overline{F(f^\circ)}\right].$$

We check that this is indeed an arrow in $\mathcal{G}(\mathbf{Y})$. Clearly, $F(f)$ has the correct domain. We check, then, that it has the correct codomain; that is, we verify that $\overline{(F(f))^\circ} = \overline{F(f^\circ)}$. By Lemma 2.0.13(iv), $(F(f))^\circ = F(f^\circ)$. It follows, then, that $\overline{(F(f))^\circ} = \overline{F(f^\circ)}$ and thus F is well defined on arrows.

Since the objects of $\mathcal{G}(\mathbf{X})$ are specific arrows in $\mathbf{X}$ and the composition in $\mathcal{G}(\mathbf{X})$ is, when defined, given by composition in $\mathbf{X}$, the functoriality of $\mathcal{G}(F)$ follows from the functoriality of F.

We check now that F is an ordered functor. That is, we must check that F preserves partial orders. Suppose that $f \leq g$ are arrows in $\mathcal{G}(\mathbf{X})$. Then $g\overline{f} = f$ and thus

$$F(g)\overline{F(f)} = F(g)F(\overline{f}) = F(g\overline{f}) = F(f).$$

Therefore, $F(f) \leq F(g)$ in $\mathcal{G}(\mathbf{Y})$ and F is an ordered functor.

Finally, we verify that F is a locally inductive functor. If $\overline{a} \wedge \overline{b}$ exists in $\mathcal{G}(\mathbf{X})$, then $\overline{a}$ and $\overline{b}$ are endomorphisms on the same object and are thus composable and in the same meet-semilattice. Then, by the functoriality of F, $F\left(\overline{a} \wedge \overline{b}\right) = F\left(\overline{a}\,\overline{b}\right) = F(\overline{a})F\left(\overline{b}\right) = F(\overline{a}) \wedge F\left(\overline{b}\right)$. □

Corollary 4.2.11. *Construction 4.2.4 is the object function of a fully faithful functor* $\mathcal{G} : \mathbf{iCat} \to \mathbf{tliGrpd}$.

Proof. By the proof of Proposition 4.2.10, $\mathcal{G}$ is clearly a faithful functor.

Let $\mathbf{X}$ and $\mathbf{X}'$ be inverse categories and suppose that $F : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{X}')$ is a locally inductive functor. We seek, then, a functor $F' : \mathbf{X} \to \mathbf{X}'$ with $\mathcal{G}(F') = F$.

For any two restriction idempotents $\overline{e}$ and $\overline{f}$ in E_A, we have $F(\overline{e} \wedge \overline{f}) = F\overline{e} \wedge F\overline{f}$ since F is locally inductive. This implies that $F\overline{e}$ and $F\overline{f}$ are $\mathbf{X}'$-endomorphisms on the same object and thus $F(E_A) \subseteq E_B$ for some object $B \in \mathbf{X}'$. So we can define, for each object $A \in \mathbf{X}$, $F'(A)$ to be the object in $\mathbf{X}'$ satisfying $F(E_A) \subseteq E_{F'(A)}$ in $\mathcal{G}(\mathbf{X}')$.

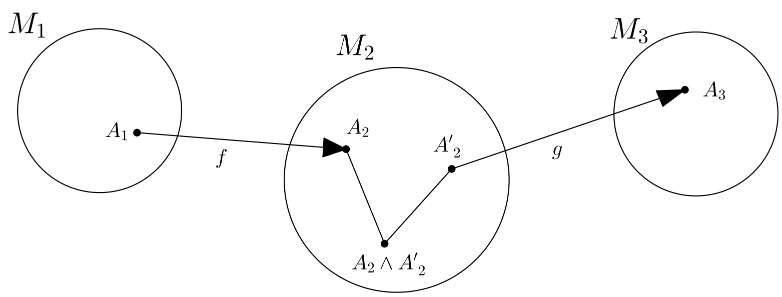
Given any arrow $f : A \to B$ in $\mathbf{X}$, we must define an arrow $F'(f) : F'(A) \to F'(B)$ in $\mathbf{X}'$. We know that f corresponds to the arrow $f : \overline{f} \to \overline{f^\circ}$ in $\mathcal{G}(\mathbf{X})$, whose image under F is $F(f) : F\overline{f} \to F\overline{f^\circ}$ in $\mathcal{G}(\mathbf{X}')$. Since $F\overline{f} \in F(E_A)$ and $F\overline{f^\circ} \in F(E_B)$, this $F(f)$ corresponds to an arrow $F'(f) : F'(A) \to F'(B)$ in $\mathbf{X}'$.

Clearly, identity arrows in $\mathbf{X}$, corresponding to identity arrows in $\mathcal{G}(\mathbf{X})$ and mapped to identities in $\mathcal{G}(\mathbf{X}')$ under F, will be mapped to identities in $\mathbf{X}'$ under F'. We check that composition is preserved. Suppose that f and g are arrows whose composite gf exists in $\mathbf{X}$. Both g and f correspond, then, to arrows $g : \overline{g} \to \overline{g^\circ}$ and $f : \overline{f} \to \overline{f^\circ}$, respectively, in $\mathcal{G}(\mathbf{X})$. Notice that the composite gf does not necessarily exist (as an arrow) in $\mathcal{G}(\mathbf{X})$, but that, since $\overline{g}, \overline{f^\circ} \in E_B$, the tensor $g \otimes f$ does and that this tensor product uniquely corresponds to gf by Proposition 4.2.8. By Proposition 4.2.9 (since F preserves meets), then, $F(g \otimes f) = F(g) \otimes F(f)$ and, again by Lemma 4.2.8 and the definition of F', corresponds to $F'(g)F'(f)$. □

Construction 4.2.12. Given a top-heavy locally inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i \in I})$, define an inverse category $\left(\mathcal{I}(\mathbf{G}), \circ, \overline{(-)}\right)$ with the following data:

- Objects: The objects are the meet-semilattices M_i.
- Arrows: $\mathcal{I}(\mathbf{G})(M_1, M_2) = \{f : A_1 \to A_2 \text{ in } \mathbf{G} \mid A_1 \in M_1,\ A_2 \in M_2\}$. Note that every object of $\mathbf{G}$ is in some M_i, and the M_i are disjoint, so that every arrow in $\mathbf{G}$ will be found in exactly one of these hom-sets.
  - Composition: A composable pair of arrows $f : M_1 \to M_2$ and $g : M_2 \to M_3$ in $\mathcal{I}(\mathbf{G})$, corresponds to a pair of arrows $f : A_1 \to A_2$ and $g : A_2' \to A_3$ in $\mathbf{G}$

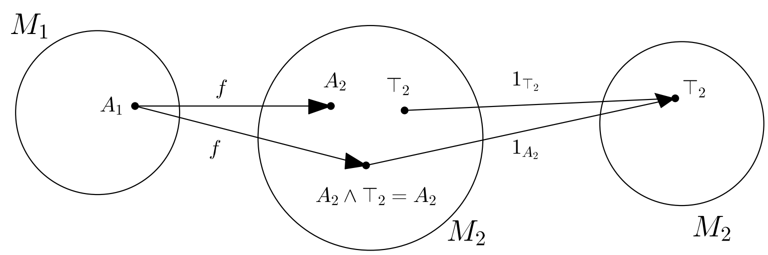
with $A_1 \in M_1$, $A_2, A_2' \in M_2$ and $A_3 \in M_3$. Since M_2 is a meet-semilattice, the meet $A_2 \wedge A_2'$ exists. We can therefore define the composite of f with g as $g \circ f = g \otimes f = [g\,|_*\, A_2 \wedge A_2'][A_2 \wedge A_2'\,{}_*|\, f]$. This composition is associative by Proposition 4.1.4.



– Identities: For each object M_1, define $1_{M_1} : M_1 \to M_1$ to be $1_{\top_1} = \top_1 \to \top_1$ in **G**. Let $f : M_1 \to M_2$ be an arrow corresponding to $f : A_1 \to A_2$ in **G**. Note that $[1_{\top_1}\,|_*\, A_1 \wedge \top_1] = 1_{A_1}$ by Proposition 4.1.5. Then

$$f \circ 1_{\top_1} = [f\,|_*\, A_1 \wedge \top_1] \bullet [A_1 \wedge \top_1\,{}_*|\, 1_{\top_1}] = [f\,|_*\, A_1] \bullet 1_{A_1} = f.$$

Similarly, $1_{\top_2} \circ f = f$.



- Restrictions: Given an arrow $f : M_1 \to M_2$ corresponding to an arrow $f : A_1 \to A_2$ in $\mathbf{G}$, define $\overline{f} : M_1 \to M_1$ by $\overline{f} = 1_{A_1} : A_1 \to A_1$. Conditions (R.1) – (R.4) saying that $\mathcal{I}(\mathbf{G})$ is a restriction category follow readily from the fact that all restriction idempotents are identities on some object in $\mathbf{G}$ and that restrictions in an ordered groupoid are unique.

- Partial Isomorphisms: For each arrow $f : M_1 \to M_2$, define $f^\circ : M_2 \to M_1$ as $f^{-1} : A_2 \to A_1$. To check that this is a restricted inverse, we check the required composites. First,

$$f \circ f^\circ = f \otimes f^\circ = [f\,|_*\, A_1 \wedge A_1] \bullet [A_1 \wedge A_1\,{}_*|\, f^{-1}] = f \bullet f^{-1} = 1_{A_2} = \overline{f^{-1}}.$$

Similarly, $f^\circ \circ f = \overline{f}$. $\diamondsuit$

Proposition 4.2.13. *For each locally inductive functor $F : \mathbf{G} \to \mathbf{H}$, there exists a functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{H})$.*

Proof. We show that F induces a functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{H})$.

Given any object in $\mathcal{I}(\mathbf{G})$, a meet-semilattice M_1, define $\mathcal{I}(F)(M_1)$ to be *the* meet-semilattice M_1' such that $F(M_1) \subseteq M_1'$. Note that this assignment of M_1' to M_1 is unique since the M_i' are a partition of $\mathbf{H}_0$.

For any arrow $f : M_1 \to M_2$ in $\mathcal{I}(\mathbf{G})$ corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$, we define $\mathcal{I}(F)(f) = F(f) : F(A_1) \to F(A_2)$, an arrow $F(f) : F(M_1) \to F(M_2)$ in $\mathcal{I}(\mathbf{G}')$. That this assignment is functorial follows from the functoriality of F. □

Corollary 4.2.14. *Construction 4.2.12 is the object function of a functor*

$$\mathcal{I} : \mathbf{tliGrpd} \to \mathbf{iCat}.$$

Proof. Let $\mathbf{G} \xrightarrow{F} \mathbf{G}' \xrightarrow{G} \mathbf{G}''$ be a composable pair of locally inductive functors. Then, on objects of $\mathcal{I}(\mathbf{G})$ (meet-semilattices forming the partition of $\mathbf{G_0}$),

$$\begin{aligned}\mathcal{I}(G)\mathcal{I}(F)(M) &= \mathcal{I}(G)(M'), \text{ where } M' \text{ such that } FM \subseteq M' \\ &= M'', \text{ where } M'' \text{ such that } M'' \supseteq G(M') = G(FM) = (GF)M \\ &= \mathcal{I}(GF)(M), \text{ by the uniqueness of } M'' \supseteq (GF)M.\end{aligned}$$

Equality of the functors $\mathcal{I}(GF)$ and $\mathcal{I}(G)\mathcal{I}(F)$ follows immediately. That $\mathcal{I}$ preserves identity functors follows from the observation that $\mathcal{I}(1_{\mathbf{G}})(M) = M$ for all objects M in $\mathcal{I}(\mathbf{G})$. □

Theorem 4.2.15. *The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence of categories,*

$$\mathbf{iCat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{tliGrpd}$$

Proof. By Corollary 4.2.11, the functor $\mathcal{G}$ is fully faithful. We show now that $\mathcal{G}$ is essentially surjective by demonstrating a natural isomorphism $\mathcal{GI} \cong 1_{\mathbf{tliGrpd}}$.

We start with a top-heavy locally inductive groupoid $(\mathbf{G}, \bullet, \leq, \{M_i\}_{i\in I})$ and we consider the composite $\mathcal{GI}(\mathbf{G})$. Recall that $\mathcal{I}(\mathbf{G})$ has as objects the meet-semilattices M_i and arrows of the form $f : M_1 \to M_2$, where $f : A_1 \to A_2$ is an arrow in $\mathbf{G}$ with $A_1 \in M_1$ and $A_2 \in M_2$. Further recall that every arrow in $\mathbf{G}$ is found exactly once in $\mathcal{I}(\mathbf{G})$. Note that for each object M_i,

$$E_{M_i} = \{\overline{f} : M_i \to M_i | f : M_i \to M_i\} = \{1_{A_i} | A_i \in M_i\} \cong M_i.$$

Then the locally inductive groupoid $\mathcal{GI}(\mathbf{G})$ contains the following data:

- Objects: $\coprod_{i\in I} E_{M_i} \cong \coprod_{i\in I} M_i = \mathbf{G}_0$.
- Arrows: For each $f : M_1 \to M_2$ in $\mathcal{I}(\mathbf{G})$ corresponding to $f : A_1 \to A_2$ in $\mathbf{G}$, there is an arrow $f : \overline{f} \to \overline{f^\circ} = f : 1_{A_1} \to 1_{A_2} \cong f : A_1 \to A_2$ in $\mathcal{GI}(\mathbf{G})$. Since arrows of $\mathbf{G}$ are appearing exactly once in $\mathcal{I}(\mathbf{G})$, we have, then, that $(\mathcal{GI}(\mathbf{G}))_1 \cong \mathbf{G}_1$.
  - Composition: Given two composable arrows corresponding to $f : A_1 \to A_2$ and $g : A_2 \to A_3$ in $\mathcal{GI}(\mathbf{G})$, we have in $\mathcal{I}(\mathbf{G})$ that

    $$g \circ f = g \otimes f = [g \,|_* \, A_2 \wedge A_2] \bullet [A_2 \wedge A_2 \, {}_*| \, f] = g \bullet f.$$

    Their composite, then, is

    $$g \star f \text{ in } \mathcal{GI}(\mathbf{G}) = g \circ f \text{ in } \mathcal{I}(\mathbf{G}) = g \bullet f \text{ in } \mathbf{G}.$$

That is, composition in $\mathcal{GI}(\mathbf{G})$ is the same as that in $\mathbf{G}$ up to isomorphism.

– Restrictions: Given an arrow $f : 1_{A_1} \to 1_{A_2} \cong f : A_1 \to A_2$ and $A_1' \leq A_1$, we have that

$$\begin{aligned}(f \mid_* A_1') \text{ in } \mathcal{GI}(\mathbf{G}) \cong f \circ 1_{A_1'} \text{ in } \mathcal{I}(\mathbf{G}) = f \otimes 1_{A_1'} \text{ in } \mathbf{G} \\ = [f \mid_* A_1 \wedge A_1'] \bullet [A_1 \wedge A_1' \mid_* 1_{A_1'}] \\ = [f \mid_* A_1'] \bullet 1_{A_1'} = [f \mid_* A_1'].\end{aligned}$$

That is, the restrictions of the two ordered groupoids $\mathbf{G}$ and $\mathcal{GI}\mathbf{G}$ are the same up to isomorphism.

This description of $\mathcal{GI}(\mathbf{G})$ is written so that the isomorphism $\mathbf{G} \cong \mathcal{GI}(\mathbf{G})$ follows immediately. □

Note. In an inverse semigroup $(S, \bullet)$, every idempotent is of the form $s^\bullet \bullet s$ for some $s \in S$. In addition, all idempotents commute. We can then consider the groupoid associated to an inverse semigroup as the Karoubi envelope of the single-object inverse category (with unit) associated to S. In a general inverse category, this fact ensures that every restriction idempotent will appear as an object in the associated top-heavy locally inductive groupoid, and that every object in this groupoid is a restriction idempotent.

The definition of the functor $\mathcal{G}$ relies on the top-heavy property of a locally inductive groupoid $\mathcal{G}$ only when defining identities on the meet-semilattices partitioning $\mathbf{G}_0$. Similarly, the identities of an inverse category $\mathbf{X}$ are essential only as top elements of the meet-semilattices E_A. In other words, removing identities from an inverse category is equivalent to removing top elements from the meet-semilattices partitioning a locally inductive groupoid. As a result, the equivalence established in Theorem 4.2.15 generalizes immediately.

Corollary 4.2.16. *The functors $\mathcal{G}$ and $\mathcal{I}$ form an equivalence*

$$\mathbf{isCat} \underset{\mathcal{I}}{\overset{\mathcal{G}}{\rightleftarrows}} \mathbf{liGrpd},$$

where **isCat** *is the category of inverse semicategories.* □

Since single-object inverse categories are precisely inverse semigroups with identity, it is clear that single-object inverse semicategories are precisely inverse semigroups. With inverse semicategories as multi-object inverse semigroups, we see that Theorem 4.1.9 – the equivalence between inductive groupoids and inverse semigroups – then follows immediately from Corollary 4.2.16.

We will end this section with a short discussion on a generalization of Theorem 4.2.15.

Recall that *prehomomorphisms* of inverse semigroups are functions between inverse semigroups satisfying $\phi(ab) \leq \phi(a)\phi(b)$. Theorem 4.1.9 can then be generalized to

Theorem 4.2.17 ([26], Theorem 8)**.** *The category of inverse semigroups and prehomomorphisms is equivalent to the category of inductive groupoids and ordered functors.*

Since the arrows of an inverse category are playing the part of "elements" in each of the "local inverse semigroups", a clear candidate for an inverse categorical analogue arises.

Definition 4.2.18. An *oplax functor* $F : \mathbf{X} \to \mathbf{X}'$ of inverse categories consists of the following data:

- for each object $A \in \mathbf{X}$, an object $F(A) \in \mathbf{X}'$;
- for each arrow $f : A \to B$, an arrow $F(f) : F(A) \to F(B)$ such that
  - for each composable pair $f : A \to B$ and $g : B \to C$ in $\mathbf{X}$, $F(gf) \leq F(g)F(f)$, and
  - for each object $A \in \mathbf{X}$, $F(1_A) \leq 1_{F(A)}$. ♢

Clearly, since composition in $\mathcal{G}(\mathbf{X})$ is defined by composition in $\mathbf{X}$, any oplax functor $F : \mathbf{X} \to \mathbf{X}'$ between inverse categories induces an ordered functor $\mathcal{G}(F) : \mathcal{G}(\mathbf{X}) \to \mathcal{G}(\mathbf{X}')$.

Suppose now that $F : \mathbf{G} \to \mathbf{G}'$ is an ordered functor between top-heavy locally inductive groupoids. Recall that composition in $\mathcal{I}(\mathbf{G})$ is defined by the tensor product in $\mathbf{G}$. Then

$$\begin{aligned}
F(g \otimes f) &= F(g\,|_* \operatorname{dom}(g) \wedge \operatorname{cod}(f))F(\operatorname{dom}(g) \wedge \operatorname{cod}(f)\,{}_*|\, f) \\
&= (Fg\,|_* F(\operatorname{dom}(g) \wedge \operatorname{cod}(f)))(F(\operatorname{dom}(g) \wedge \operatorname{cod}(f))\,{}_*|\, Ff) \\
&\leq (Fg\,|_* F(\operatorname{dom}(g) \wedge F\operatorname{cod}(f))(F\operatorname{dom}(g) \wedge F\operatorname{cod}(f)\,{}_*|\, Ff) \\
&= Fg \otimes Ff
\end{aligned}$$

and thus F induces an oplax functor $\mathcal{I}(F) : \mathcal{I}(\mathbf{G}) \to \mathcal{I}(\mathbf{G}')$. Specifically, since the identities in $\mathcal{I}(\mathbf{G})$ are the top elements of $\mathbf{G}$, $\mathcal{I}(F)$ is strict on identities.

These arguments can then be easily extended to prove the following.

Theorem 4.2.19. *The category of top-heavy locally inductive groupoids and ordered functors is equivalent to the category of inverse categories and oplax functors.* □

4.3 Etendues, Sheaves and Morphisms

In this section, we equip inverse categories with structure which encodes topological data. Equipping inverse categories with joins allows us to "glue" *compatible* morphisms together and equipping their corresponding top-heavy locally inductive groupoid given by Theorem 4.2.15 with Ehresmann topologies allow us to take sheaves on these groupoids. We take the construction of sheaves on ordered groupoids with Ehresmann topologies given by [28] and study it in context of our top-heavy locally inductive groupoids. We will also introduce a suitable notion of morphism for ordered groupoids equipped with Ehresmann topologies.

Join Inverse Categories

Definition 4.3.1 ([26])**.** Let $\mathbf{X}$ be a restriction category. Two arrows f and g in $\mathbf{X}$ are *compatible* – denoted $f \smile g$ – if and only if $f\overline{g} = g\overline{f}$. A subset $S \subseteq \mathbf{X}_1$ of arrows in $\mathbf{X}$ is called a *compatible set* whenever every pair of arrows in S is compatible. ♢

Definition 4.3.2 ([9])**.** A *join restriction category* is a restriction category in which for every compatible set $(f_i : A \to B)_{i \in I}$, there is a map $\bigvee_{i\in I} f_i : A \to B$ such that

(i) for all $i \in I$, $f_i \leq \bigvee_{i\in I} f_i$,

(ii) if there exists a map g such that $f_i \leq g$ for all $i \in I$, then $\bigvee f_i \leq g$,

(iii) for any $h : B \to C$, $h\left(\bigvee_{i\in I} f_i\right) = \bigvee_{i\in I} hf_i$.

Note that these facts follow:

(i) for any $j \in I$, $\overline{f_j}\left(\bigvee_{i \in I} f_i\right) = f_j$,

(ii) for any $h : C \to A$, $\left(\bigvee_{i \in I} f_i\right) h = \bigvee_{i \in I} f_i h$,

(iii) $\overline{\bigvee_{i \in I} f_i} = \bigvee_{i \in I} \overline{f_i}$. ♢

Definition 4.3.3. A morphism of join inverse categories is a functor preserving all joins. The category of join inverse categories and join-preserving functors is denoted **jiCat**. ♢

We consider a join inverse category $\mathbf{X}$. Clearly, since $\mathbf{X}$ is an inverse category with an additional property, we can apply the functor $\mathcal{G}$ to $\mathbf{X}$ to obtain a top-heavy locally inductive groupoid. We study now what properties the groupoid $\mathcal{G}(\mathbf{X})$ inherits from the join structure on $\mathbf{X}$.

Definition 4.3.4. An order ideal A of a poset $(P, \leq)$ is a subset $A \subseteq P$ with the following two properties:

(i) For every $x \in A$, if $y \leq x$ then $y \in A$; A is down-closed.

(ii) For every $x, y \in A$, there exists $z \in A$ such that $x \leq z$ and $y \leq z$; A is a directed set. ♢

Definition 4.3.5. For each object $\overline{f}$ in $\mathcal{G}(\mathbf{X})$, the *principal order ideal* of $\overline{f}$ is the set of objects

$$\downarrow \overline{f} = \left\{ \overline{e} \in \mathcal{G}(\mathbf{X})_0 : \overline{e} \leq \overline{f} \right\}. \quad ♢$$

The following proposition is a direct consequence of the definition of compatible set along with the commutativity of restrictions.

Proposition 4.3.6. *For each object $\overline{f} \in \mathcal{G}(\mathbf{X})$, the principal order ideal $\downarrow \overline{f}$ is a compatible set.*

Proposition 4.3.7. *Let $\mathbf{X}$ be a join inverse category. For each object $\overline{f} \in \mathcal{G}(\mathbf{X})$, the principal order ideal $\downarrow \overline{f}$ is a locale with all joins inherited from $\mathbf{X}$ and meet defined by $\overline{a} \wedge \overline{b} = \overline{a}\overline{b}$.*

Proof. Together with the partial order and joins inherited from $\mathbf{X}$, $\downarrow \overline{f}$ is a poset with all joins by Proposition 4.3.6. That $\downarrow \overline{f}$ has finite meets is analogous to the proof of Proposition 4.2.2. Finally, in $\downarrow \overline{f}$, finite meets distribute over arbitrary joins since joins come from $\mathbf{X}$, composition distributes over joins and our meet is defined by composition. □

Proposition 4.3.8. *Let $\mathbf{X}$ be a join inverse category. For each arrow $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ in $\mathcal{G}(\mathbf{X})$, there is a frame homomorphism $\alpha_* :\downarrow \overline{\alpha} \to\downarrow \overline{\alpha^\circ}$ defined by $\alpha_*\left(\overline{b}\right) = \overline{(\overline{b}\,\alpha^\circ)}$.*

Proof. We first verify that α_* is well defined: for any $\overline{b} \in\downarrow \overline{\alpha}$,

$$\overline{\alpha^\circ}\left(\overline{\overline{(\overline{b}\,\alpha^\circ)}}\right) = \overline{\alpha^\circ}\overline{(\overline{b}\alpha^\circ)} = \overline{\alpha^\circ}\,\overline{(b\,\alpha^\circ)} = \overline{(b\,\alpha^\circ)} = \overline{(\overline{b}\alpha^\circ)}$$

and thus $\overline{(\overline{b}\,\alpha^\circ)} \leq \overline{\alpha^\circ}$ and $\alpha_*\left(\overline{b}\right) \in\downarrow \overline{\alpha^\circ}$.

Second, we verify that α_* preserves the partial order: suppose that $\overline{b} \leq \overline{c}$ in $\downarrow \overline{\alpha}$.

Then

$$\begin{aligned}
\alpha_*(\overline{c})\overline{\left(\alpha_*(\overline{b})\right)} &= \overline{(\overline{c}\,\alpha^\circ)}\,\overline{\overline{(\overline{b}\,\alpha^\circ)_B}} = \overline{(\overline{c}\,\alpha^\circ)}\,\overline{(\overline{b}\,\alpha^\circ)} \\
&= \overline{(c\,\alpha^\circ)}\,\overline{(b\,\alpha^\circ)} = \overline{\left(c\,\alpha^\circ\,\overline{(b\,\alpha^\circ)}\right)} \\
&= \overline{\left(c\,\overline{b}\,\alpha^\circ\right)} = \overline{\left(\overline{c}\,\overline{b}\,\alpha^\circ\right)} \\
&= \overline{(\overline{b}\,\alpha^\circ)} = \alpha_*\left(\overline{b}\right)
\end{aligned}$$

and thus $\alpha_*\left(\overline{b}\right) \leq \alpha_*(\overline{c})$ in $\downarrow \overline{\alpha^\circ}$.

Thirdly, we check that α_* preserves finite meets:

$$\begin{aligned}
\alpha_*\left(\overline{b}\right) \wedge \alpha_*(\overline{c}) &= \alpha_*\left(\overline{b}\right)\alpha_*(\overline{c}) \\
&= \overline{(\overline{b}\,\alpha^\circ)}\,\overline{(\overline{c}\,\alpha^\circ)} = \overline{(b\alpha^\circ)}\,\overline{(c\alpha^\circ)} \\
&= \overline{\left(b\,\alpha^\circ\overline{(c\,\alpha^\circ)}\right)} = \overline{(b\,\overline{c}\,\alpha^\circ)} \\
&= \overline{\left(\overline{b}\,\overline{c}\,\alpha^\circ\right)} = \alpha_*\left(\overline{b}\,\overline{c}\right) \\
&= \alpha_*\left(\overline{b} \wedge \overline{c}\right)
\end{aligned}$$

Finally, we check that α_* preserves arbitrary joins: suppose that $S \subseteq \downarrow \overline{\alpha}$ is any subset of $\downarrow \overline{\alpha}$. Since S is a compatible set and the join in $\downarrow \overline{\alpha}$ is inherited from $\mathbf{X}$,

$$\alpha_*\left(\bigvee_{\overline{s}\in S}\overline{s}\right) = \overline{\left(\left(\bigvee_{\overline{s}\in S}\overline{s}\right)\alpha^\circ\right)} = \overline{\left(\bigvee_{\overline{s}\in S}\overline{s}\alpha^\circ\right)} = \bigvee_{\overline{s}\in S}\overline{(\overline{s}\,\alpha^\circ)} = \bigvee_{\overline{s}\in S}\alpha_*(\overline{s}) \qquad \square$$

Corollary 4.3.9. *Let* $\mathbf{X}$ *be a join inverse category. There is a contravariant functor*

$$(-)_* : \mathcal{G}(\mathbf{X})^{\mathrm{op}} \to \mathbf{Loc},$$

where **Loc** *is the category of locales and locale morphisms.*

Proof. We show that $(-)_*$ is a covariant functor $\mathcal{G}(\mathbf{X}) \to \mathbf{Frm}$, where **Frm** is the category of frames and frame homomorphisms.

The object function $\overline{f} \mapsto \left(\overline{f_A}\right)_* = \downarrow \overline{f}$ is well defined by Lemma 4.3.7 and for each $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$, the arrow function $\alpha \mapsto \alpha_*$ is a frame homomorphism by Lemma 4.3.8.

This arrow function preserves identities: for all objects $\overline{f}$ and $\overline{c_A} \in \downarrow \overline{f}$,

$$\left(1_{\overline{f}}\right)_* (\overline{c}) = \left(\overline{f}\right)_* (\overline{c}) = \overline{\left(\overline{c}\left(\overline{f}\right)^\circ\right)} = \overline{c}\,\overline{f} = \overline{c} = 1_{\downarrow \overline{f}} (\overline{c})\,.$$

This arrow function also preserves composition: given any composable pair $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ and $\beta : \overline{\beta} \to \overline{\beta^\circ}$ of arrows (i.e., $\overline{\beta} = \overline{\alpha^\circ}$) and any object $\overline{c} \in \downarrow \overline{\alpha}$,

$$(\beta\alpha)_*(\overline{c}) = \overline{(\overline{c}\,(\beta\alpha)^\circ)} = \overline{(\overline{c}\,\alpha^\circ\beta^\circ)} = \overline{\left(\overline{(\overline{c}\,\alpha^\circ)}\beta^\circ\right)} = \overline{(\alpha_*(\overline{c})\,\beta^\circ)} = (\beta_*\alpha_*)(\overline{c}). \qquad \square$$

We may also define a contravariant version of the frame homomorphism in Lemma 4.3.8. The following Lemma and its corresponding Corollary can be proved exactly as were Lemma 4.3.8 and Corollary 4.3.9.

Lemma 4.3.10. *Let* **X** *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is a frame homomorphism* $\alpha^* : \downarrow \overline{\alpha^\circ} \to \downarrow \overline{\alpha}$ *defined by* $\alpha^* (\overline{e}) = \overline{(\overline{e}\,\alpha)}$. $\square$

Corollary 4.3.11. *Let* **X** *be a join inverse category. There is a covariant functor*

$$(-)^* : \mathcal{G}(\mathbf{X}) \to \mathbf{Loc},$$

where **Loc** *is the category of locales and locale morphisms.* $\square$

One would expect from our choice of notation for the frame homomorphisms in Lemmas 4.3.8 and 4.3.10 that, for each arrow α in $\mathcal{G}(\mathbf{X})$, that there is an adjunction $\alpha^* \dashv \alpha_*$. Note that, for all $\overline{e_A} \in \downarrow \overline{\alpha_A}$, we have that

$$\alpha^*\alpha_*(\overline{e}) = \alpha^*\overline{(\overline{e}\,\alpha^\circ)} = \overline{\left(\overline{(\overline{e}\,\alpha^\circ)}\,\alpha\right)} = \overline{(\overline{e}\,\alpha^\circ\,\alpha)} = \overline{e}\,\overline{\alpha} = \overline{e}.$$

Similarly, for all $\overline{f} \in \downarrow \overline{\alpha^\circ}$, we have $\alpha_*\alpha^*\overline{f} = \overline{f}$. That is, there is indeed an adjunction $\alpha^* \dashv \alpha_*$, whose unit and counit are natural isomorphisms.

Theorem 4.3.12. *Let* $\mathbf{X}$ *be a join inverse category. For each arrow* $\alpha : \overline{\alpha} \to \overline{\alpha^\circ}$ *in* $\mathcal{G}(\mathbf{X})$, *there is an equivalence of categories*

$$\downarrow \overline{\alpha} \underset{\alpha^*}{\overset{\alpha_*}{\rightleftarrows}} \downarrow \overline{\alpha^\circ}$$

□

We end this subsection with a class of examples of join inverse categories: complete inverse semigroups.

Definition 4.3.13 ([26])**.** Two elements s and t of an inverse semigroup are *compatible* whenever both st^{-1} and $s^{-1}t$ are idempotents of S. A subset A of an inverse semigroup S is *compatible* whenever any pair of elements in A is compatible. ♢

Definition 4.3.14 ([26])**.** An inverse semigroup S is said to be *complete* whenever every non-empty compatible subset of S has a join. ♢

Proposition 4.3.15. *Complete inverse semigroups with identity are single-object join inverse categories.*

Proof. Let $S = Hom(*, *)$ be the hom-set of a single-object join inverse category. Then S is an inverse semigroup with identity. We show that the compatibility relations coincide.

Let $f, g \in S$ and first suppose that $f\overline{g} = g\overline{f}$. Then $fg^\circ fg^\circ = f\overline{g}g^\circ f\overline{g}g^\circ = gg^\circ g\overline{f}g^\circ = f\overline{g}g^\circ = fg^\circ$ and thus $fg^\circ \in E(S)$. Similarly, $f^\circ g \in E(S)$; f and g are compatible in the inverse semigroup sense.

Conversely, suppose that both $f^\circ g$ and $fg^\circ \in E(S)$. We claim that $f \wedge g$ exists and that $f \wedge g = f\overline{g}$. This argument follows closely that given in [Lemma 13, Lawson 1998]. Note first that $f\overline{g} = fg^\circ g \leq f, g$. Suppose that w is such that $w \leq f, g$. Then $\overline{w} \leq \overline{g}$ and thus $w = w\overline{w} \leq f\overline{w} \leq f\overline{g}$. Therefore, $f \wedge g = f\overline{g}$. Since fg° is an idempotent, $f^\circ g = (fg^\circ)^\circ = g^\circ f$ is also an idempotent and, by symmetry, $f \wedge g = g\overline{f}$. Then $f\overline{g} = f \wedge g = g\overline{f}$; f and g are compatible in the restriction category sense. □

The Ehresmann Site Associated to a Join Inverse Category

Definition 4.3.16. Define a relation $\leq_{\mathcal{J}}$ on the objects of an ordered groupoid by $a \leq_{\mathcal{J}} b$ if and only if there exists an object $a' \cong a$ such that $a' \leq b$. That is, a is isomorphic to some object sitting below b. ♢

Definition 4.3.17 ([28])**.** Let $(\mathbf{G}, \circ, \leq)$ be an ordered groupoid. An *Ehresmann topology* on $\mathbf{G}$ is an assignment of, for each object $e \in \mathbf{G}$, a collection $T(e)$ of order ideals of $\downarrow e$ – called *covering ideals* – satisfying

(i) $\downarrow e \in T(e)$ for each object $e \in \mathbf{G}$.

(ii) Let e and f be objects of $\mathbf{G}$ such that $f \leq_{\mathcal{J}} e$. Then for each $x : f \cong e' \leq e$ and $\mathcal{A} \in T(e)$, we have $x^{-1} \otimes \mathcal{A} \otimes x \in T(f)$.

(iii) Let e be an object of $\mathbf{G}$, let $\mathcal{A} \in T(e)$ and let $\mathcal{B} \trianglelefteq \downarrow e$ be an arbitrary order ideal of $\downarrow e$. Suppose that, for each $x : f \cong e' \leq e$ (where $e' \in \mathcal{A}$), we have $x^{-1} \otimes \mathcal{B} \otimes x \in T(f)$. Then $\mathcal{B} \in T(e)$.

An ordered groupoid equipped with an Ehresmann topology is an *Ehresmann site*. $\diamondsuit$

Notation. We denote the category of Ehresmann sites and ordered functors as **ESite**.

The following theorem gives an explicit relationship between a (locally localic) join inverse category and the topological data attached to its corresponding groupoid.

Theorem 4.3.18. *If* $\mathbf{X}$ *is a join inverse category, then* $\mathcal{G}(\mathbf{X})$ *admits an Ehresmann site with, for each object* $\overline{e} \in \mathcal{G}(\mathbf{X})$,

$$T(\overline{e}) = \left\{ \mathcal{S} \trianglelefteq \downarrow \overline{e} : \bigvee \mathcal{S} = \overline{e} \right\}.$$

Proof. We check that each of the three conditions required of T being an Ehresmann topology are satisfied:

(i) Since $\downarrow \overline{e}$ is principal, it has top element $\bigvee \downarrow \overline{e} = \overline{e}$ and thus $\downarrow \overline{e} \in T(\overline{e})$.

(ii) Let $\overline{e}$ and $\overline{f}$ be objects of $\mathcal{G}(\mathbf{X})$ such that $\overline{f} \leq_{\mathcal{J}} \overline{e}$, let $\alpha : \overline{f_B} \to \overline{e'} \leq \overline{e}$ be arbitrary and let $\mathcal{A} \in T(\overline{e})$. Notice that, for any object $\overline{x} \in \mathcal{G}(X)$, $\overline{x} = \mathrm{id}_{\overline{x}} : \overline{x} \to \overline{x}$ so that $\mathrm{dom}(\overline{x}) = \mathrm{cod}(\overline{x}) = \overline{x}$. Then

$$\begin{aligned}\bigvee_{\overline{a} \in \mathcal{A}} \alpha^\circ \otimes \overline{a} \otimes \alpha &= \bigvee_{\overline{a} \in \mathcal{A}} \alpha^\circ \otimes \left(\overline{a}\, \alpha \right) = \bigvee_{\overline{a} \in \mathcal{A}} \alpha^\circ\, \overline{a}\, \alpha = \alpha^\circ \left(\bigvee_{\overline{a} \in \mathcal{A}} \overline{a} \right) \alpha \\ &= \alpha^\circ\, \overline{e}\, \alpha = \alpha^\circ\, \overline{\alpha^\circ}\, \overline{e}\, \alpha = \alpha^\circ\, \overline{e'}\, \overline{e}\, \alpha = \alpha^\circ\, \overline{e'}\, \alpha \\ &= \alpha^\circ\, \overline{\alpha^\circ}\, \alpha = \alpha^\circ\, \alpha = \overline{f}\end{aligned}$$

and thus $\alpha^\circ \otimes \mathcal{A} \otimes \alpha \in T\left(\overline{f}\right)$.

(iii) Let $\overline{e}$ be an object of $\mathcal{G}(\mathbf{X})$, let $\mathcal{A} \in T(\overline{e})$ and let $\mathcal{B} \trianglelefteq \downarrow \overline{e}$ be an arbitrary order ideal of $\downarrow \overline{e}$. Suppose also that, for all arrows $\alpha : \overline{f} \to \overline{e'} \leq \overline{e}$ with $\overline{e'} \in \mathcal{A}$, we have that $\alpha^\circ \otimes \mathcal{B} \otimes \alpha \in T\left(\overline{f}\right)$. That is, for all such α,

$$\overline{f} = \bigvee \alpha^\circ \otimes \mathcal{B} \otimes \alpha = \alpha^\circ \left(\bigvee \mathcal{B}\right) \alpha$$

and thus

$$\overline{\alpha^\circ} \left(\bigvee \mathcal{B}\right) \overline{\alpha^\circ} = \alpha \, \overline{f} \, \alpha^\circ = \alpha \, \alpha^\circ = \overline{\alpha^\circ}.$$

In particular, each identity map $\overline{a} : \overline{a} \to \overline{a} \leq \overline{e}$, where $\overline{a} \in \mathcal{A}$, is such a map and

$$\overline{a} \left(\bigvee \mathcal{B}\right) \overline{a} = \overline{a}.$$

Since $\bigvee \mathcal{B}$ is a restriction map with domain X and $\bigvee \mathcal{B} \leq \overline{e}$, we see that $\bigvee \mathcal{B}$ commutes with every restriction map with domain X and $\overline{e} \bigvee \mathcal{B} = \bigvee \mathcal{B}$. Then for all $\overline{a}$, $\overline{a} \left(\bigvee \mathcal{B}\right) \overline{a} = \overline{a} \bigvee \mathcal{B}$ and thus

$$\overline{e} = \bigvee_{\overline{a} \in \mathcal{A}} \overline{a} = \bigvee_{\overline{a} \in \mathcal{A}} \overline{a} \left(\bigvee \mathcal{B}\right) = \left(\bigvee_{\overline{a} \in \mathcal{A}} \overline{a}\right) \bigvee \mathcal{B} = \overline{e} \bigvee \mathcal{B} = \bigvee \mathcal{B}$$

and $\mathcal{B} \in T(\overline{e})$. □

We can now define a functor $\mathcal{G}_E : \mathbf{jiCat} \to \mathbf{ESite}$ defined by assigning, for each join inverse category $\mathbf{X}$, the above Ehresmann topology to the ordered groupoid $\mathcal{G}(\mathbf{X})$.

Join Inverse Categories from Ehresmann Sites

While Ehresmann topologies can be constructed from inverse categories with joins, there is no clear reason that an Ehresmann topology will imply the existence of joins, at least coherently (coherent in the sense that the joins in $\mathbf{X}$ are the same as those in the Ehresmann topology admitted by $\mathcal{G}(\mathbf{X})$). Consequently, the functor $\mathcal{G}_E$ seems not to be part of an equivalence of categories between **jiCat** and **ESite**. We may seek, however, a left or right adjoint to $\mathcal{G}_E$ which, in some way, is given by the equivalence established by $\mathcal{G}$ and $\mathcal{I}$ in Theorem 4.2.15.

This adjoint functor can be discovered by investigating the image of a join inverse category under $\mathcal{G}$ and asking: where do the joins land and how can we get back from them?

4.4 The Etendue of Sheaves on an Ehresmann Site

Etendues are toposes which are "locally like a space" (in a sense that will be explained below). Etendues can be presented as sheaves on a left-cancellative site [25]. Lawson and Steinberg use this presentation to formally interpret the theory of étendues in the language of ordered categories [28]. Funk has also shown that étendues can be interpreted via the classifying topos of an inverse semigroup [16].

It seems natural, then to consider join inverse categories and ask how the topological data given in Section 4.3 can be used to extend Lawson and Steinberg's work to the context of join inverse categories (and locally inductive Ehresmann sites).

Ehresmann Sites from Left Cancellative Sites

As noted, most definitions, results and details for this subsection can be found in [27, 28]. Note, however, that we have translated all statements from Lawson's "category as a set with a partial monoid operation" language into the "categories as objects and arrows" language to better suit this thesis.

Definition 4.4.1. A *left cancellative category* is a category in which every morphism is monic. The category of left cancellative categories and functors is denoted **lcCat**. ♢

Construction 4.4.2 ([27], Left Cancellative Categories to Ordered Groupoids)**.** A span (f, f') in $\mathbf{C}$ is a pair of co-initial arrows f and f'. We say that two spans (f, f') and (g, g') are isomorphic whenever there is a $\mathbf{C}$-isomorphism $u : \mathrm{dom}(f) \to \mathrm{dom}(g)$ with $gu = f$ and $g'u = f'$. Given a left cancellative category $\mathbf{C}$, define an ordered groupoid $\mathcal{G}_C(\mathbf{C})$ with the following data:

- Objects: Isomorphism classes of spans of the form

  $$[f, f] = \left[B \xleftarrow{\;f\;} A \xrightarrow{\;f\;} B \right]$$

  in $\mathbf{C}$.

- Arrows: an arrow $[f, f'] : [f, f] \to [f', f']$ is an isomorphism class of spans

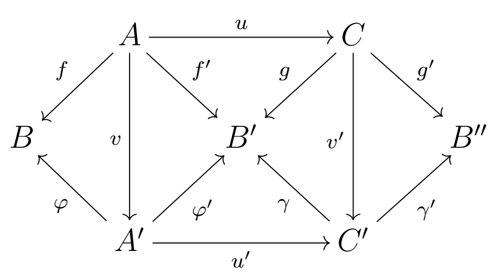
  $$[f, f'] = \left[B \xleftarrow{\;f\;} A \xrightarrow{\;f'\;} B' \right]$$

  in $\mathbf{C}$. Note that these arrows are not defined as traditional maps between spans are.

– Composition: If $\mathrm{dom}[g, g'] = \mathrm{cod}[f, f']$, then there exists an isomorphism u in $\mathbf{C}$ such that $gu = f'$. We then define the composite $[g, g'][f, f']$ to be the isomorphism class

$$[f, g'u] = \left[B \xleftarrow{\;f\;} A \xrightarrow{\;g'u\;} B'' \right]$$

For completeness, we show that this composition is well defined (proof of this is omitted in [27]). Suppose that $[f, f'] = [\varphi, \varphi']$ and $[g, g'] = [\gamma, \gamma']$ are arrows in $\mathcal{G}_C(\mathbf{C})$ with $\mathrm{dom}[g, g'] = \mathrm{cod}[f, f']$ and $\mathrm{dom}[\gamma, \gamma'] = \mathrm{cod}[\varphi, \varphi']$. That is, there are $\mathbf{C}$-isomorphisms $v : \mathrm{dom}(f) \to \mathrm{dom}(\varphi)$, $v' : \mathrm{dom}(g) \to \mathrm{dom}(\gamma)$, $u : \mathrm{dom}(f) \to \mathrm{dom}(g)$ and $u' : \mathrm{dom}(\varphi) \to \mathrm{dom}(\gamma)$ such that the following diagram is commutative:



We demonstrate the equality $[f, g'u] = [g, g'][f, f'] = [\gamma, \gamma'][\varphi, \varphi'] = [\varphi, \gamma'u']$, then, by giving a $\mathbf{C}$-isomorphism $s : A \to A'$ satisfying $\varphi s = f$ and $\gamma'u's = g'u$. We claim that v, immediately satisfying the first condition, is such an isomorphism. To verify the second condition, we first note that

$$\varphi'v = f' = gu = \gamma v'u = \varphi'(u')^{-1}v'u.$$

Therefore, since **C** is left cancellative, $v = (u')^{-1}v'u$ and thus

$$\gamma' u' v = \gamma' u'(u')^{-1} v' u = \gamma' v' u = g' u.$$

– Identities: We define $1_{[f,f]} = [f, f]$. Since $\mathrm{cod}[f, f] = [f, f] = \mathrm{dom}[f, f']$, the isomorphism $u : \mathrm{dom}(f) \to \mathrm{dom}[f]$ is the identity and we get that

$$[f, f'][f, f] = [f, f' 1_{\mathrm{dom}(f)}] = [f, f' 1_{\mathrm{dom}(f')}] = [f, f'].$$

Similarly, $[f', f'][f, f'] = [f, f']$.

– Inverses: Given an arrow $[f, f']$, we define $[f, f']^{-1} = [f', f]$. Then the required identities $[f, f'][f', f] = [f', f']$ and $[f', f][f, f'] = [f, f]$ are easily seen to hold.

- Partial order: We define a partial order on the arrows by $[f, f'] \leq [g, g']$ if and only if $(f, f') = (g, g')p = (gp, g'p)$ for some arrow $p \in \mathbf{C}$. ♢

This construction turns out to be functorial with respect to the following assignment: a functor $F : \mathbf{C} \to \mathbf{D}$ of left cancellative categories is assigned to an ordered functor

$$\mathcal{G}_C(F) : \mathcal{G}_C(\mathbf{C}) \to \mathcal{G}_C(\mathbf{D})$$

defined by $\mathcal{G}_C(F)[f, f'] = [Ff, Ff']$.

Theorem 4.4.3 ([27], Theorem 2.3)**.** *The construction $\mathcal{G}_C$ is the object function of a* *weakly essentially surjective functor*

$$\mathcal{G}_C : \mathbf{lcCat} \to \mathbf{oGrpd}.$$

□

Question. Is this functor part of a biequivalence of bicategories?

Definition 4.4.4. A *sieve* on an object e in a category $\mathbf{C}$ is a collection of morphisms with codomain e that are closed under precomposition with morphisms in $\mathbf{X}$. ♢

Definition 4.4.5. A *Grothendieck topology* on a category $\mathbf{C}$ is a function J assigning to each object e a collection $J(e)$ of sieves (called *covering sieves*) satisfying the following three conditions:

(i) (Identity) $e\mathbf{C} = \{f \in \mathbf{C} : \mathrm{cod}(f) = e\} \in J(e)$.

(ii) (Change of Base) If $f : e \to e'$ is an arrow in $\mathbf{C}$ and $S \in J(e')$, then

$$f^*S = \{g \in eC : fg \in S\} \in J(e).$$

(iii) (Local Character) If $S \in J(e')$ and R is any sieve on e' such that $f^*S \in J(e)$ for all $f : e \to e'$, then $R \in J(e')$. ♢

Definition 4.4.6. A category $\mathbf{C}$ equipped with a Grothendieck topology is called a *site*. If all arrows in $\mathbf{C}$ are monomorphisms, we call $\mathbf{C}$ a *left cancellative site*. ♢

Definition 4.4.7. A morphism $F : (\mathbf{C}, J) \to (\mathbf{C}', J')$ of sites is a functor $F : \mathbf{C} \to \mathbf{C}'$ satisfying the following two conditions:

(i) (F is covering-preserving) If $\mathcal{U} \in J(e)$ then $F\mathcal{U} = \{FU \,|\, U \in \mathcal{U}\} \in J'(Fe)$.

(ii) (F is covering-flat) For each diagram $D : \mathbf{I} \to \mathbf{C}$ and each cone $\varepsilon : e \Rightarrow FD$, the sieve

$$\mathcal{T}_\varepsilon = \{v_t \to e \,|\, t : u \to v_t \text{ in } \mathbf{C}' \text{ such that } \exists \text{ cone } \eta : e' \Rightarrow D$$
$$\text{in } \mathbf{C} \text{ with } F\eta.t' = \varepsilon.t \text{ for some } t' : e \to e'_t \to Fe' \text{ in } \mathbf{C}'\}$$

is a covering sieve in $J'(e)$. ♢

Comparing the definition of a Grothendieck topology to that of an Ehresmann topology, we see that the two are quite similar if we substitute the conjugation of A by x with the sieve x^*A. Indeed, there are bijective correspondences between Grothendieck topologies on left-cancellative categories and Ehresmann topologies on ordered groupoids. These correspondences are worked out in detail in [28] and we present those results here for our use.

Theorem 4.4.8 ([28], Lemma 3.2 and Theorem 3.5)**.** *Let* $\mathbf{C}$ *be a left cancellative category and* e *an object of* $\mathbf{C}$.

(a) *(i) Let* S *be a sieve on* e *in* $\mathbf{C}$. *Then*

$$[S] = \{[f, f] : f \in S\}$$

is an order ideal of $\downarrow [e, e]$.

(ii) If J is a Grothendieck topology on $\mathbf{C}$, then

$$T[e,e] = \{[S] : S \in J\}$$

defines an Ehresmann topology on $\mathcal{G}_C(\mathbf{C})$.

(b) (i) Let A be an order ideal of $\downarrow [e,e]$. Then

$$||A|| = \{f \in \mathbf{C} : [f,f] \in A\}$$

is a sieve on e in $\mathbf{C}$.

(ii) If T is an Ehresmann topology on $\mathcal{G}_C(\mathbf{C})$, then

$$J(e) = \{||A|| : A \in T[e,e]\}$$

defines a Grothendieck topology on $\mathbf{C}$.

(c) The operations $S \mapsto [S]$ and $A \mapsto ||A||$ are mutually inverse; there is a bijection between the sets of sieves on e in $\mathbf{C}$ and the set of order ideals of $\downarrow [e,e]$ in $\mathcal{G}_C(\mathbf{C})$.

(d) There is a bijective correspondence between Grothendieck topologies on $\mathbf{C}$ and Ehresmann topologies on $\mathcal{G}_C(\mathbf{C})$.

Left Cancellative Sites from Ehresmann Sites

Construction 4.4.9 ([28], Ordered Groupoids to Left Cancellative Categories)**.** Given an Ehresmann site $(\mathbf{G}, T)$, define a left cancellative category $\mathcal{L}_C(\mathbf{G}, T)$ with the following data:

- Objects: Same objects as $\mathbf{G}$.
- Arrows: An arrow $f : A \to B$ is an arrow f from $\mathbf{G}$ with $\mathrm{dom}(f) = A$ and $\mathrm{cod}(f) \leq B$:

$$\begin{array}{ccc} & & B \\ & & \vee| \\ A & \xrightarrow{f} & B' \end{array}$$

  - Composition: Consider two composable arrows:

$$\begin{array}{ccccc} & & & & C \\ & & & & \vee| \\ & & B & \xrightarrow{g} & C' \\ & & \vee| & & \\ A & \xrightarrow{f} & B' & & \end{array}$$

    We define their composite as $gf = g \otimes f = [g\,|_*\,\mathrm{cod}(f)]f : A \to C$.

  - The identities are given by the identities in $\mathbf{G}$. ♢

This construction turns out to be functorial with respect to the following assignment: an ordered functor $F : \mathbf{G} \to \mathbf{H}$ is assigned to a functor

$$\mathcal{L}_C(F) : \mathcal{L}_C(\mathbf{G}) \to \mathcal{L}_C(\mathbf{H})$$

defined by $\mathcal{L}_C(F)(f) = Ff$.

Theorem 4.4.10 ([27], Theorem 2.3)**.** *The construction $\mathcal{L}_C$ is the object function of a functor $\mathcal{L}_C$:* **oGrpd** $\to$ **lcCat**.

Theorem 4.4.11 ([28], Lemma 3.6 and Theorem 3.8)**.** *Let* $\mathbf{G}$ *be an ordered groupoid and e an object of* $\mathbf{G}$.

(a) *(i) Let A be an order ideal of $\downarrow e$ in $\mathbf{G}$. Then*

$$A^\flat = (\{e\} \times A)\mathbf{G}$$

is a sieve on (e,e) in $\mathcal{L}_C(\mathbf{G})$.

(ii) Let T be an Ehresmann topology on $\mathbf{G}$. Then

$$J(e,e) = \{A^\flat : A \in T(e)\}$$

defines a Grothendieck topology on $\mathcal{L}_C(\mathbf{G})$.

(b) *(i) Let S be a sieve on (e,e) in $\mathcal{L}_C(\mathbf{G})$. Then*

$$S^\sharp = \{f : (e,f) \in S\}$$

is an order ideal of $\downarrow e$ in $\mathbf{G}$.

(ii) Let J be a Grothendieck topology on $\mathcal{L}_C(\mathbf{G})$. Then

$$T(e) = \{S^\sharp : S \in J(e,e)\}$$

defines an Ehresmann topology on $\mathbf{G}$.

(c) The operations $S \mapsto S^\sharp$ and $A \mapsto A^\flat$ are mutually inverse; there is a bijection between the sets of sieves on (e,e) in $\mathcal{L}_C(\mathbf{G})$ and the set of order ideals of $\downarrow e$ in $\mathbf{G}$.

(d) There is a bijective correspondence between Ehresmann topologies on $\mathbf{G}$ and Groth-

endieck topologies on $\mathcal{L}_C(\mathbf{G})$.

Having now reviewed the necessary structures from Lawson and Steinberg, we introduce the structures required to define a suitable morphism of Ehresmann sites. The intuition underlying the following definitions stems from thinking of the tensor product as the composition (which will be guaranteed to exist in any (locally) inductive groupoid).

Definition 4.4.12. A $\otimes$-*cone* ε of a diagram $D : \mathbf{I} \to \mathbf{G}$ in an ordered groupoid $\mathbf{G}$ with vertex $e \in \mathbf{G_0}$ – denoted $\varepsilon : e \otimes\!\!\Rightarrow D$ – is a collection $\{\varepsilon_i : e \to e_i \leq D_i\}$ of $\mathbf{G}$-isomorphisms satisfying, for all $f_{ij} : D_i \to D_j$ in $\mathbf{I}$, $f_{ij} \otimes \varepsilon_i = \varepsilon_j$. ♢

Note. Observe that if ε is a $\otimes$-cone in $\mathbf{G}$, then ε is a (strict) cone in $\mathcal{L}_C(\mathbf{G})$.

Definition 4.4.13. A morphism $F : (\mathbf{G}, T) \to (\mathbf{G}', T')$ of Ehresmann sites is an ordered functor $F : \mathbf{G} \to \mathbf{G}'$ satisfying the following two conditions:

(i) (F is ideal-preserving) If $\mathcal{A} \in T(e)$ then $F\mathcal{A} = \{Fa \mid a \in \mathcal{A}\} \in T'(Fe)$.

(ii) (F is ideally-flat) For each diagram $D : \mathbf{I} \to \mathbf{G}$ and each $\otimes$-cone $\varepsilon : e \otimes\!\!\Rightarrow FD$, the order ideal

$$\mathcal{Q}_\varepsilon = \{v_t \leq e \mid \exists t : u \to v_t \text{ in } \mathbf{G}' \text{ such that } \exists \otimes\text{-cone } \eta : e' \otimes\!\!\Rightarrow D$$
$$\text{in } \mathbf{G} \text{ with } F\eta \otimes t' = \varepsilon \otimes t \text{ for some } t' : e \to e'_t \leq Fe' \text{ in } \mathbf{G}'\}$$

is a (covering) ideal in $T(e)$. ♢

Proposition 4.4.14. *Let* $\mathbf{G}$ *be an ordered groupoid. For each diagram* $D : \mathbf{I} \to \mathbf{G}$ *and each* $\otimes$*-cone* $\varepsilon : e \otimes\!\!\Rightarrow FD$, $\mathcal{Q}_\varepsilon$ *as defined above is indeed an order ideal of* $\downarrow e$.

Proof. Clearly, $\mathcal{Q}_\varepsilon \subseteq \downarrow e$. We now show that it is down-closed. Suppose that $v_t \in \mathcal{Q}_\varepsilon$ and $v \leq v_t$. Then there is a $\otimes$-cone $\eta : e' \otimes\!\!\Rightarrow D$ in $\mathbf{G}$ with $F\eta \otimes t' = \varepsilon \otimes t$ for some $t' : e \to e'_t \leq Fe'$ in $\mathbf{G}'$. The following diagram will be helpful in reading through this argument:

$$\begin{array}{ccccc}
 & & e & \xrightarrow{\varepsilon_i} & e_i \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
u & \xrightarrow{t} & v_t & \xrightarrow{[\varepsilon_i \,|_*\, v_t]} & B \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
A & \xrightarrow{[v_* |\, t]} & v & \xrightarrow{[\varepsilon_i \,|_*\, v]} & C
\end{array}$$

We claim that the $\otimes$-cone η guaranteed to exist by v_t still functions as desired with v when we consider $[t' \,|_*\, A]$ in the place of t', which exists since $A \leq \mathrm{dom}(t) = \mathrm{dom}(t')$. We first note that

$$F\eta \otimes [t' \,|_*\, A] = [F\eta \otimes t' \,|_*\, A] = [\varepsilon \otimes t \,|_*\, A].$$

Now, for each $i \in \mathbf{I}$, we have

$$\begin{aligned}
[\varepsilon_i \otimes t \,|_*\, A] &= \Big[[\varepsilon_i \,|_*\, v_t] \bullet t \,|_*\, A\Big] \\
&= \Big[[\varepsilon_i \,|_*\, v_t] \,|_*\, \mathrm{cod}[t \,|_*\, A]\Big] \bullet [t \,|_*\, A] \\
&= \Big[[\varepsilon_i \,|_*\, v_t] \,|_*\, v\Big] \bullet [v \,|_*\, t] \\
&= [\varepsilon_i \,|_*\, v] \bullet [v \,|_*\, t] \\
&= \varepsilon_i \otimes [v \,|_*\, t]
\end{aligned}$$

and thus $F\eta \otimes [t' \,|_*\, A] = [\varepsilon \otimes t \,|_*\, A] = \varepsilon \otimes [v \,|_*\, t]$. Since $\mathrm{cod}[v \,|_*\, t] = v \leq e$, we have $v \in \mathcal{Q}_\varepsilon$ and $\mathcal{Q}_\varepsilon$ is an order ideal of $\downarrow e$. $\square$

Theorem 4.4.15. *If* $F : (\mathbf{C}, J) \to (\mathbf{C}', J')$ *is a map of left cancellative sites (i.e., is both covering-preserving and covering-flat), then*

$$\mathcal{G}_C(F) : (\mathcal{G}_C(\mathbf{C}), \mathcal{G}_C(J)) \to (\mathcal{G}_C(\mathbf{C}'), \mathcal{G}_C(J'))$$

is a morphism of Ehresmann sites.

Proof. We first prove that $\mathcal{G}_C(F)$ is ideal-preserving. Suppose that $[1_e, 1_e] \in \mathcal{G}_C(\mathbf{C})_0$ and $\mathcal{A} \in \mathcal{G}_C(J)[1_e, 1_e]$. Then $e \in \mathbf{C}_0$ and $\mathcal{A} = [S]$ for some $S \in J(e)$. Since F is covering-preserving, then, $FS \in J'(Fe)$. Therefore, $\mathcal{G}_C(F)(\mathcal{A}) = \mathcal{G}_C(F)[S] = [FS] \in \mathcal{G}_C(J')[F1_e, F1_e] = \mathcal{G}_C(J')(F[1_e, 1_e])$ and $\mathcal{G}_C(F)$ is ideal-preserving.

We now prove that $\mathcal{G}_C(F)$ is ideally-flat. Suppose that $D : \mathbf{I} \to \mathcal{G}_C(\mathbf{C})$ is a diagram in $\mathcal{G}_C(\mathbf{C})$ and that $\varepsilon : [e, e] \otimes\!\!\Rightarrow \mathcal{G}_C(F)(D)$ is a tensor cone over $\mathcal{G}_C(F)(D)$ in $\mathcal{G}_C(\mathbf{C}')$. Let $\mathcal{Q}_\varepsilon$ be defined as above. Explicitly,

$$\begin{aligned}\mathcal{Q}_\varepsilon = \{[v_t, v_t] \leq [e, e] \mid\ & [t, v_t] : [t, t] \to [v_t, v_t] \text{ in } \mathcal{G}_C(\mathbf{C}') \\ & \text{such that } \exists \otimes\text{-cone } \eta : [e', e'] \otimes\!\!\Rightarrow D \text{ in } \mathcal{G}_C(\mathbf{C}) \\ & \text{with } \mathcal{G}_C(F)(\eta) \otimes [t', e'_t] = \varepsilon \otimes [t, v_t] \text{ for some } [t', e'_t]\}\end{aligned}$$

Recall that $\otimes$-cones in $\mathcal{G}_C(\mathbf{C}')$ are cones in $\mathcal{L}_C(\mathcal{G}_C(\mathbf{C}')$ and composition in $\mathcal{L}_C(\mathcal{G}_C(\mathbf{C}'))$ is given by $\otimes$ in $\mathcal{G}_C(\mathbf{C}')$. It is then immediate by the definition of $\mathcal{L}_C$ on ordered functors that

$$\mathcal{Q}_\varepsilon^\flat = \{[t, v_t] : [t, t] \to [v_t, v_t] \,|\, v_t \in \mathcal{Q}_\varepsilon\}$$

is a sieve satisfying the properties required in Definition 4.4.7(ii). Note also that

$\mathcal{L}_C(\mathcal{G}_C(F)) \simeq F$. Since F is covering-flat, then,

$$\mathcal{Q}_\varepsilon^\flat \in \mathcal{L}_C(\mathcal{G}_C(J))[e,e]$$

and thus $(\mathcal{Q}_\varepsilon^\flat)^\sharp = \mathcal{Q}_\varepsilon \in \mathcal{G}_C(J)[e,e]$ by Theorem 4.4.11. □

4.5 Future Work

As previously mentioned, we note that Construction **??** gives a complete inverse semigroup. Considering ideally flat and ideally covering functors as the morphisms between Ehresmann sites, one may wonder if there exists a canonical choice of such functors given an functor between inverse categories. After having defined the arrow part of the functors $\mathcal{I}_E$ and $\mathcal{G}_E$, we may ask

Question. Does $\mathcal{I}_E(\mathbf{G}, T)$ define the object function of a left adjoint $\mathcal{I}_E \dashv \mathcal{G}_E$?

We may also wonder if it is possible to generalize this adjunction to include more than complete inverse semigroups.

Question. Is there some information encoded in the Ehresmann topology T of an Ehresmann site $(\mathbf{G}, T)$ that allows us to interpret $\mathbf{G}$ (freely or otherwise) as only *locally* inductive, so that $\mathcal{I}_E(\mathbf{G}, T)$ is a multi-object inverse semicategory. If so, are there top-heavy locally inductive groupoids so that we can construct join inverse categories?

Maps Between the Topoi of Sheaves on Ehresmann Sites

Theorem 4.4.15 implies that ideally flat and ideally covering functors between Ehresmann sites correspond to geometric morphisms between their corresponding topoi of sheaves. In the construction of some double category whose objects are Ehresmann sites, these ideally flat and covering functors will comprise the vertical category. The horizontal category, then, is up for discussion.

Our future approach will use "generalized maps" between Ehresmann sites, namely "special" modules. What is to be meant here by special is not yet clear, but it is likely that we will want these modules to correspond to "interesting" maps between the corresponding topoi of sheaves, be these geometric morphisms or otherwise. We will now explore the choice of putting ordinary modules between the Ehresmann sites, in the sense that they are ordinary modules between the underlying groupoids (as of now, no interaction with the Ehresmann topologies on these groupoids seems to be needed).

Question. Consider a module $M : (\mathbf{G}, T) \longrightarrow\!\!\!\!\!\!\!\mapsto (\mathbf{G}', T')$. How should one define a functor

$$M_* : \mathrm{Sh}(\mathbf{G}, T) \to \mathrm{Sh}(\mathbf{G}', T')?$$

Sheaves on Ehresmann Sites

Definition 4.5.1. Consider $\mathbf{G}_0$ as a posetal category. A *presheaf* on an Ehresmann site $(\mathbf{G}, T)$ is a contravariant functor $F : \mathbf{G}_0^{\mathrm{op}} \to \mathbf{Set}$. A *matching family* $\{a_i\}_{i \in A}$ for an order ideal $A \in T(e)$ is a family of $a_i \in F(i)$ for each $i \in A$ with the following property: if $j \leq i$, then $F(j \leq i)(a_i) = a_j$. An *amalgamation* of a matching family

$\{a_i\}_{i \in A}$ for an order ideal $A \in T(e)$ is an element $a \in F(e)$ such that $F(i \leq e)(a) = a_i$ for all $i \in A$. A presheaf is called a *sheaf* if every matching family has a unique amalgamation. ♢

A topos is a category which can be thought of as a generalized space [32]. Etendues were originally defined by Grothendieck in [2, Ex. 9.8.2] as topoi that are "locally like spaces" in the following sense:

Definition 4.5.2 ([2])**.** An *etendue* is a topos $\mathcal{T}$ with an object $E \in \mathcal{T}$ such that $! : E \to 1$ is an epimorphism and the slice category $\mathcal{T}/E$ is equivalent to the category of sheaves on a topological space. ♢

The following two theorems allow us to think of étendues from the perspective of ordered groupoids.

Theorem 4.5.3 ([28], Theorem 3.10)**.** *Every étendue is presented by a site constructed from an ordered groupoid equipped with an Ehresmann topology.*

Theorem 4.5.4 ([28], Theorem 4.5)**.** *Every étendue is equivalent to the topos of sheaves on an Ehresmann site.*

Conjecture 4.5.5. The equivalence in Theorem 4.5.4 induces a weakly essentially surjective functor $\mathrm{Sh}_{ES} : \mathbf{ESite} \to \mathbf{Et}$.

Question. Should Conjecture 4.5.5 be true, is this functor part of a biequivalence?

Sheaves on Join Inverse Categories

By Lemma 4.3.7, we know that a join inverse category is locally localic. Since there is a notion of sheaves on a locale and there is some relationship between join inverse categories and Ehresmann sites, the following question naturally arises.

Question. What do sheaves on a join inverse category look like? In particular, what information do we get for free about sheaves on its corresponding Ehresmann site?

Since join inverse categories have a corresponding groupoid and are locally localic, we make the following conjecture:

Conjecture 4.5.6. Every topos of sheaves on a localic groupoid is equivalent to the topos of sheaves on some join inverse category.

Chapter 5: Two-Dimensional Restriction Categories

This chapter introduces double restriction categories and restriction bicategories as settings for the study of two-dimensional restriction categories. Double restriction categories are data structures with two compatible restriction structures, while restriction bicategories are such that the restriction structure on the 1-cells extends functorially to its 2-cells.

We define restriction category objects so that restriction categories internal to **Set** are small restriction categories and therefore restriction categories internal to **rCat** are double restriction categories. We show in general that categories internal to a suitable **C** can be viewed as restriction monads internal to Span(**C**).

Restriction bicategories are bicategories equipped with a restriction operator on its 1-cells which is functorial in that it extends to a vertical restriction structure on its 2-cells. We give as a motivating examples the restriction bicategory of restriction bimodules, and that restriction **Cat**-categories are restriction (strict) bicategories (i.e., strict 2-categories).

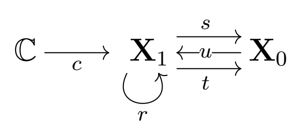
Finally, these two structures together are used to define a double restriction category whose horizontal bicategory is given by (supported range) restriction bimodules and whose (total) vertical category is given by (restriction) functors. This double category provides a convenient setting in which we conjecture a generalization of the Ehresmann-Schein-Nambooripad Theorem.

5.1 Double Categories

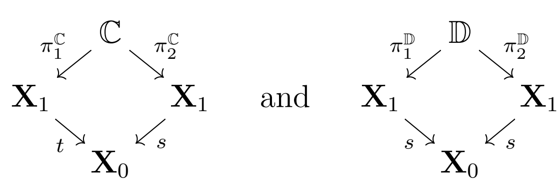
Internal Restriction Categories

This section defines restriction categories diagrammatically in such a way that (ordinary) restriction categories are restriction categories internal to **Set**. The motivation for this definition, and consequently the assurance that it is complete, is the fact that monads in Span(**C**) (for a category **C** with enough pullbacks) can be interpreted as categories internal to **C**. This definition, then, is such that restriction monads in Span(**C**) can be interpreted as restriction categories internal to **C**.

Definition 5.1.1. A restriction category (in **Set**) contains the following data:



$\mathbb{C}$ and $\mathbb{D}$ denote the pullbacks



We require that

(i) $s.u = 1$ and $t.u = 1$,

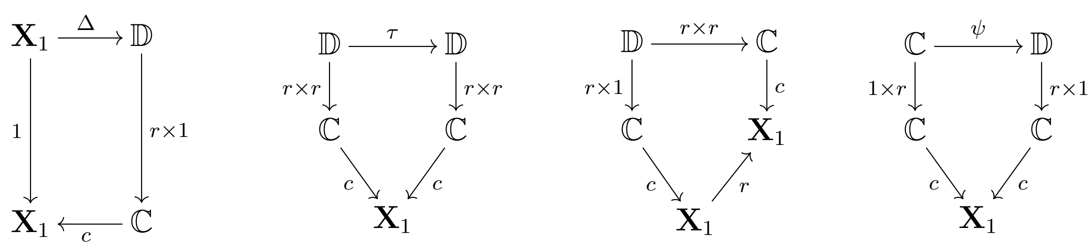
(ii) $s.c = s.\pi_1$ and $t.c = t.\pi_2$,

(iii) (assoc.) $c.(1 \times c) = c.(c \times 1)$,

(iv) (unit) $c.(u \times 1) = \pi_2$ and $c.(1 \times u) = \pi_1$,

(v) (rest.) $sr = s = tr$,

and the following diagrams commute (where Δ and τ are the usual diagonal and canonical flip in **Set**) :



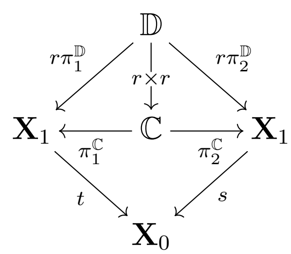
(R.1) (R.2) (R.3) (R.4)

◊

One can quickly verify that

- In (R1): $(f, g) \in \mathbb{D}$ implies that $(rf, g) \in \mathbb{C}$, or that $r \times 1 : \mathbb{D} \to \mathbb{C}$ is a function;
- In (R2): $(f, g) \in \mathbb{D}$ implies that $(rf, rg) \in \mathbb{C}$, or that $r \times r : \mathbb{D} \to \mathbb{C}$ is a function; and
- In (R4): $(f, g) \in \mathbb{C}$ implies that $(f, rg) \in \mathbb{C}$, or that $1 \times r : \mathbb{C} \to \mathbb{C}$ is a function.

Consider now that we take such objects internal to any category with pullbacks over s and t. We should check, then, that the maps used in the conditions (R.1) through (R.4) can actually be defined. For example, the map $r \times r : \mathbb{D} \to \mathbb{C}$ is defined by the universal property of the pullback $(\mathbb{C}, \pi_1^{\mathbb{C}}, \pi_2^{\mathbb{C}})$ over $\mathbf{X}_1 \xrightarrow[t]{} \mathbf{X}_0 \xleftarrow[s]{} \mathbf{X}_1$, over which $(\mathbb{D}, r\pi_1^{\mathbb{D}}, r\pi_2^{\mathbb{D}})$ is also a cone:

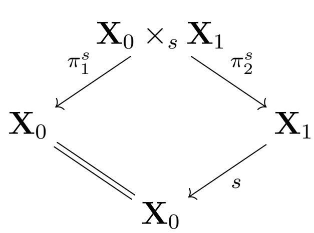


In this case, that $(\mathbb{D}, r\pi_1^{\mathbb{D}}, r\pi_2^{\mathbb{D}})$ is a cone is easy to see since

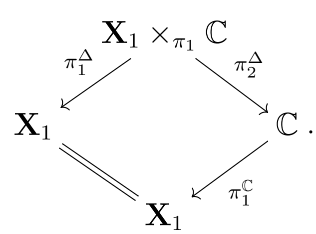
$$tr\pi_1^{\mathbb{D}} = s\pi_1^{\mathbb{D}} = s\pi_2^{\mathbb{D}} = sr\pi_1^{\mathbb{D}}.$$

We have no qualms naming this universal map $r \times r$ since this is how it (by universality, it is unique) behaves if our category of residence is **Set**.

Define $\mathbf{X}_0 \times_s \mathbf{X}_1$ and $\mathbf{X}_1 \times_{\pi_1} \mathbb{C}$ by the pullback squares



and



Over the cospan $\mathbf{X}_1 \xrightarrow[t]{} \mathbf{X}_0 \xleftarrow[s]{} \mathbf{X}_1$, we define the maps

- $1 \times r : \mathbb{C} \to \mathbb{C}$ as the universal arrow induced by the cone $(\mathbb{C}, \pi_1^{\mathbb{C}}, r\pi_2^{\mathbb{C}})$,
- $r \times 1 : \mathbb{D} \to \mathbb{C}$ as the universal arrow induced by the cone $(\mathbb{D}, r\pi_1^{\mathbb{D}}, \pi_2^{\mathbb{D}})$,

- $u \times 1 : \mathbf{X}_0 \times_s \mathbf{X}_1 \to \mathbb{C}$ as the universal arrow induced by the cone $(\mathbf{X}_0 \times_s \mathbf{X}_1, u\pi_1^s, \pi_2^s)$;

over $\mathbf{X}_1 \xrightarrow[s]{} \mathbf{X}_0 \xleftarrow[s]{} \mathbf{X}_1$, the maps

- $\Delta : \mathbf{X}_1 \to \mathbb{D}$ as the universal arrow induced by the cone $(\mathbf{X}_1, 1, 1)$,
- $\tau : \mathbb{D} \to \mathbb{D}$ as the universal arrow induced by the cone $(\mathbb{D}, \pi_2^{\mathbb{D}}, r\pi_1^{\mathbb{D}})$,
- $\psi : \mathbb{C} \to \mathbb{D}$ as the universal arrow induced by the cone $(\mathbb{C}, c, \pi_1^{\mathbb{C}})$,
- $\widetilde{\psi} : \mathbf{X}_1 \times_{\pi_1} \mathbb{C} \to \mathbb{D}$ as the universal arrow induced by the cone $(\mathbf{X}_1 \times_{\pi_1} \mathbb{C}, \pi_1^{\Delta}, c\pi_2^{\Delta})$;

over $\mathbf{X}_0 = \mathbf{X}_0 \xleftarrow[s]{} \mathbf{X}_1$, the map

- $1 \times r : \mathbf{X}_0 \times_s \mathbf{X}_1 \to \mathbf{X}_0 \times_s \mathbf{X}_1$ as the universal arrow induced by the cone $(\mathbf{X}_0 \times_s \mathbf{X}_1, \pi_1^s, r\pi_2^s)$;

and over $\mathbf{X}_1 = \mathbf{X}_1 \xleftarrow[\pi_1^{\mathbb{C}}]{} \mathbb{C}$, the map

- $\Delta \times 1 : \mathbb{C} \to \mathbf{X}_1 \times_{\pi_1} \mathbb{C}$ as the universal arrow induced by the cone $(\mathbb{C}, \pi_1^{\mathbb{C}}, r\pi_2^{\mathbb{C}})$.

A restriction category internal to **Set** is a small restriction category with $\overline{f} = rf$. As such, a restriction category internal to **Set** can be used as a model to encode Proposition 2.0.3 in the language of internal categories.

Proposition 5.1.2. *A restriction category internal to a category with pullbacks satisfies the following equations:*

(i) $c.(r \times r).\Delta = r$;

(ii) $c.(r \times r).\tau.\widetilde{\psi}.(\Delta \times 1) = r.c$;

(iii) $r.c.(1 \times r) = r.c$;

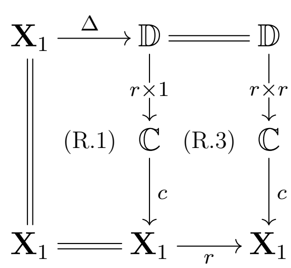
(iv) $r.r = r$;

(v) $r.c.(r \times r) = c.(r \times r)$;

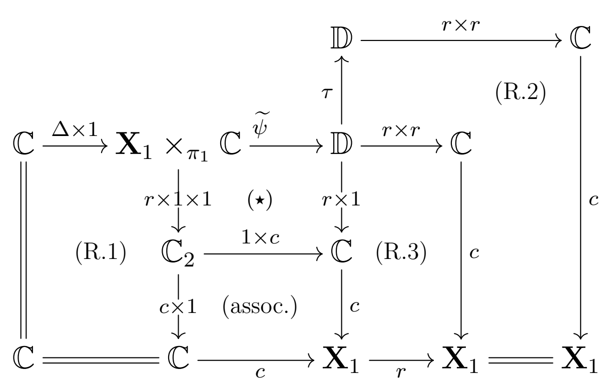
(vii) *If* $c.(r \times 1) = \pi_2$, *then* $r.\pi_2 = \pi_2.(1 \times r)$.

Proof. The statements follow from the commutativity of the following diagrams, where the individual cells commute by the rules indicated in parentheses:

(i)



(ii)



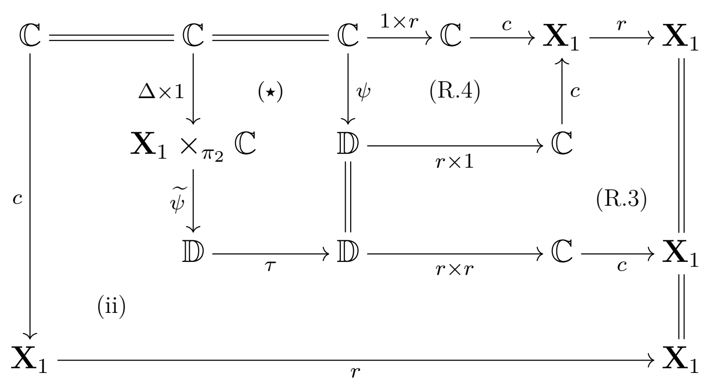
The cell $(\star)$ commutes since

$$\pi_1.(r \times 1).\widetilde{\psi} = r.\pi_1.\widetilde{\psi} = r.\pi_1 = \pi_1.(r \times 1 \times 1) = \pi_1.(1 \times c).(r \times 1 \times 1)$$

and

$$\pi_2.(r \times 1).\widetilde{\psi} = \pi_2.\widetilde{\psi} = c.\pi_2 = c.\pi_2.(r \times 1 \times 1) = \pi_2.(1 \times c).(r \times 1 \times 1).$$

(iii)



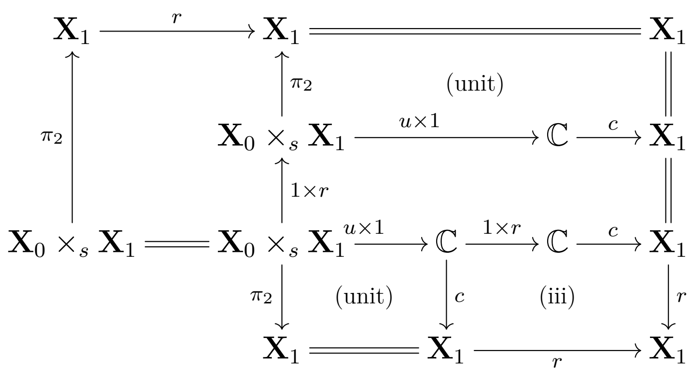
The cell $(\star)$ commutes since

$$\pi_1.\tau.\widetilde{\psi}.(\Delta \times 1) = \pi_2.\widetilde{\psi}.(\Delta \times 1) = c.\pi_2.(\Delta \times 1) = c.\pi_2 = \pi_1.\psi$$

and

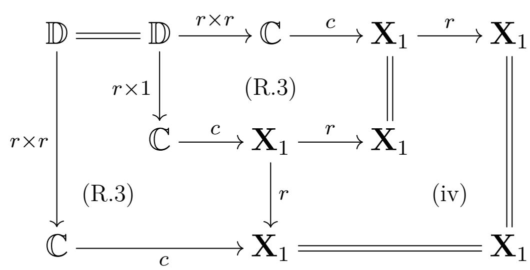
$$\pi_2.\tau.\widetilde{\psi}.(\Delta \times 1) = \pi_1.\widetilde{\psi}.(\Delta \times 1) = \pi_1.(\Delta \times 1) = \pi_1 = \pi_2.\psi.$$

(iv) The commutativity of the diagram

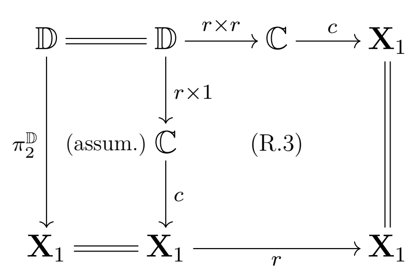


implies that $r.\pi_2 = r.r.\pi_2$. Since $(\mathbf{X}_0 \times_s \mathbf{X}_1, \pi_1, \pi_2)$ is the pullback of the cospan $(1, \mathbf{X}_0, s)$, π_2 is monic and therefore $r = r.r$.

(v)



(vii)



□

Proposition 3.1.2 shows that restriction monads internal to **Set** are precisely small restriction categories. This correspondence can be extended to restriction categories internal to any suitable category $\mathbf{C}$.

Theorem 5.1.3. *Suppose that* $\mathbf{C}$ *is a category with all pullbacks over* s *and* t*, and all coproducts (in particular, a terminal object). Suppose also that the terminal object* 1 *is a generator for* $\mathbf{C}$*. Restriction categories internal to* $\mathbf{C}$ *correspond to restriction monads in* $\mathrm{Span}(\mathbf{C})$.

Proof. Given a restriction monad

$$T = \quad \mathbf{X}_0 \xleftarrow{\;s\;} \mathbf{X}_1 \xrightarrow{\;t\;} \mathbf{X}_0$$

in $\mathrm{Span}(\mathbf{C})$, the underlying monad (T, η, μ) corresponds to an ordinary category internal to $\mathbf{C}$. We now define the restriction map r. Following the discussion in Chapter 3.1, recall that the "objects" of $\mathbf{X}$ are morphisms of the form $A : 1 \to \mathbf{X}_0$ and each is encoded by a span

$$\vec{A} = \quad \{*\} \xleftarrow{\;\mathrm{id}\;} \{*\} \xrightarrow{\;A\;} \mathbf{X}_0 \qquad .$$

Additionally, each "arrow" of $\mathbf{X}$ is a morphism $f : 1 \to \mathbf{X}_1$ and fits into some diagram

$$\begin{array}{ccccc} & & 1 & & \\ & \swarrow^{\mathrm{id}} & \downarrow f & \searrow^{\vec{B}} & \\ 1 & & & & \mathbf{X}_0 \\ & \nwarrow_{\pi_1} & & \nearrow_{\mathbf{X}_0} & \\ & & 1 \;{}_A\!\times_s \mathbf{X}_1 & & \end{array}$$

which is a span morphism in $\mathrm{Span}(\mathbf{C})(1, \mathbf{X}_0)(\vec{B}, T\vec{A})$ and encodes f with source A

and target B. We can then identify the coproduct of all such hom-sets with $\mathbf{X}_1$ and use ρ to define r as the composite

$$\mathbf{X}_1 \cong \coprod_A \coprod_B \mathrm{Span}(\mathbf{C})(1, \mathbf{X}_0)(\vec{B}, T\vec{A}) \xrightarrow{\rho} \coprod_A \mathrm{Span}(\mathbf{C})(1, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

Similarly, given a restriction category internal to $\mathbb{C}$, one defines ρ locally using r. $\square$

Double Restriction Categories

Internalizing the definition of a restriction category results immediately in a coherent way of generalizing restriction categories to two dimensions using double categories, allowing us to simultaneously keep track of two interacting restriction structures.

Definition 5.1.4. A double restriction category is a restriction category internal to **rCat**. $\diamondsuit$

It is well known that double categories, seen as category objects internal to **Cat**, are equivalent to double categories interpreted as a suitably structured collection of objects, horizontal arrows, vertical arrows and double cells. A similar description for double restriction categories facilitates reasoning about double restriction categories in a graphical way.

Definition 5.1.5. A double restriction category is a double category

$$\mathbf{D} = (\mathrm{Obj}(\mathbf{D}), \mathrm{Ver}(\mathbf{D}), \mathrm{Hor}(\mathbf{D}), \mathrm{Dbl}(\mathbf{D}))$$

such that

- The horizontal category $(\mathrm{Ver}(\mathbf{D}), \mathrm{Dbl}(\mathbf{D}))$ is equipped with a restriction operator $\overline{(-)}$ which is an assignment of double cells

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle v} \\ C & \xrightarrow[g]{} & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{\overline{f}} & A \\ {\scriptstyle u}\downarrow\!\bullet & \overline{\alpha} & \bullet\!\downarrow{\scriptstyle u} \\ C & \xrightarrow[\overline{g}]{} & C \end{array}$$

  which satisfies (R1) through (R4).

- The vertical category $(\mathrm{Hor}(\mathbf{D}), \mathrm{Dbl}(\mathbf{D}))$ is equipped with a restriction operator $\widetilde{(-)}$ which is an assignment of double cells

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow\!\bullet & \alpha & \bullet\!\downarrow{\scriptstyle v} \\ C & \xrightarrow[g]{} & D \end{array} \quad \longmapsto \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \widetilde{u}}\downarrow & \widetilde{\alpha} & \downarrow{\scriptstyle \widetilde{v}} \\ A & \xrightarrow[f]{} & B \end{array}$$

  which satisfies (R1) through (R4).

- The restriction operations commute: $\overline{(-)} \circ \widetilde{(-)} = \widetilde{(-)} \circ \overline{(-)}$. $\diamondsuit$

That Definitions 5.1.4 and 5.1.5 are equivalent can be easily verified using the intuition that

- $(\mathbf{X}_0)_0$ corresponds to $\mathrm{Obj}(\mathbf{D})$,
- $(\mathbf{X}_0)_1$ corresponds to $\mathrm{Hor}(\mathbf{D})$,
- $(\mathbf{X}_1)_0$ corresponds to $\mathrm{Ver}(\mathbf{D})$, and
- $(\mathbf{X}_1)_1$ corresponds to $\mathrm{Dbl}(\mathbf{D})$.

The horizontal restriction category structure $(\mathrm{Ver}(\mathbf{D}), \mathrm{Dbl}(\mathbf{D}))$ will then correspond to the restriction category structure of $\mathbf{X}_1$ and the vertical restriction category structure $(\mathrm{Hor}(\mathbf{D}), \mathrm{Dbl}(\mathbf{D}))$ will correspond to the internal restriction category (i.e., that whose composition is c and whose restriction will be defined by r).

The restriction structure in $\mathbf{X}_1$ and the functoriality of r (together with $sr = tr = s$) ensures that a typical double cell α in a double restriction category then has restriction endocells of the appropriate type:

$$\begin{array}{ccc} A & \xrightarrow{\overline{f}} & A \\ {\scriptstyle u}\big\downarrow\!\bullet & \overline{\alpha} & \bullet\!\big\downarrow{\scriptstyle u} \\ C & \xrightarrow[\overline{g}]{} & C \end{array} \quad \overset{\overline{(-)}}{\Longleftarrow} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\big\downarrow\!\bullet & \alpha & \bullet\!\big\downarrow{\scriptstyle v} \\ C & \xrightarrow[g]{} & D \end{array} \quad \overset{\widetilde{(-)}}{\Longrightarrow} \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \widetilde{u}}\big\downarrow & \widetilde{\alpha} & \big\downarrow{\scriptstyle \widetilde{v}} \\ A & \xrightarrow[f]{} & B \end{array}$$

Finally, that r is in particular a restriction functor on $\mathbf{X}_1$ means that the vertical and horizontal restriction structures commute:

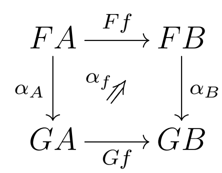
$$\overline{\widetilde{\alpha}} = \overline{r\alpha} = r\overline{\alpha} = \widetilde{\overline{\alpha}}$$

Limits in Restriction Categories

We briefly review restricted limits in restriction categories, which will allow us to define two examples of double restriction categories.

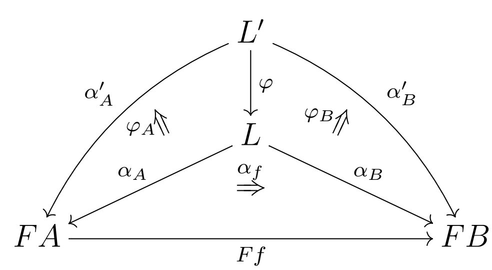
Definition 5.1.6. Suppose that $F, G : \mathbf{C} \rightrightarrows \mathbf{D}$ are 2-functors. A *lax natural transformation* $\alpha : F \rightsquigarrow G$ is a family $(\alpha_A : FA \to GA)_{A \in \mathbf{C}}$ of 1-cells in $\mathbf{D}$, indexed by 0-cells in $\mathbf{C}$, such that, for each 1-cell $f : A \to B$ in $\mathbf{C}$, there is 2-cell $\alpha_f : (Gf)\alpha_A \Rightarrow \alpha_B(Ff)$

in **D** :



◇

Definition 5.1.7. Let $F : \mathbf{J} \to \mathbf{C}$ be a diagram in the 2-category $\mathbf{C}$. Let L be a 0-cell of $\mathbf{C}$ and let $L : \mathbf{J} \to \mathbf{C}$ be the constant 2-functor on L; that is, all 0-cells are sent to L and all 1- and 2-cells are sent to identities. A *lax cone* over the diagram F is a lax natural transformation $\alpha : L \rightsquigarrow F$. We call a lax cone a *universal lax cone* whenever it has the following universal property: if $\alpha' : L' \rightsquigarrow F$ is any other lax cone over F, there is a unique 1-cell $\varphi : L' \to L$ in $\mathbf{C}$ such that, for all 1-cells $f : A \to B$ in $\mathbf{J}$, there exist 2-cells $\varphi_A : \alpha_A\varphi \Rightarrow \alpha'_A$ and $\varphi_B : \alpha_B\varphi \Rightarrow \alpha'_B$ in $\mathbf{C}$ such that $\alpha'_f.\varphi_A = \varphi_B.\alpha_f\varphi$. That is, these 2-cells fit into the following diagram in the appropriate way (imagine the 2-cell α'_f running along the front face of this tetrahedron):



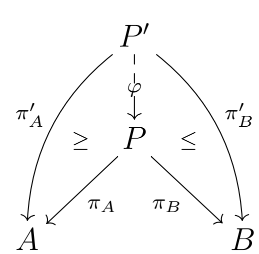
A *lax limit* L of a 2-diagram F is a universal lax cone $\alpha : L \rightsquigarrow F$. ◇

Definition 5.1.8 ([10])**.** Let $\mathbf{X}$ be a restriction category and $\mathbf{J}$ an ordinary category. A *restricted limit* of a diagram $F : \mathbf{J} \to \mathbf{X}$ in $\mathbf{X}$ is a universal lax cone $\alpha : L \rightsquigarrow F$ over F such that each component α_A is total. In addition, if $\alpha' : L' \rightsquigarrow F$ is another lax cone (whose components need not be total) over F, then the universal arrow $\varphi : L' \to L$ has restriction $\overline{\varphi} = \overline{\alpha'_{FA}}\,\overline{\alpha'_{FB}}\,\overline{\alpha'_{FC}}\,\ldots$, the meet of restrictions of the components of α'. ♢

We make a few comments on the intuition of this definition. First, that we have the factorization through the universal arrow be only up to 2-cell (which, in the 2-category **rCat**, are $\leq$ given by the restriction order) corresponds with our intuition that "taking a detour" through the universal arrow has the possibility of being less defined than the "direct route" from L' to any of the objects in our diagram. Second, that the restriction idempotent corresponding to the universal arrow is the meet of the restriction idempotents of the lax cone over our diagram corresponds to our intuition that this universal arrow should be the "most possibly defined" arrow having this universal property.

Example 5.1.9. Here are some examples of restricted limits in a restriction category.

(a) Restricted products in a restriction category $\mathbf{X}$: Given any two objects A and B in $\mathbf{X}$, a restricted product is a cone consisting of an object P and total arrows $\pi_A : P \to A$ and $\pi_B : P \to B$ satisfying the following universal property: for each (lax) cone (P', π'_A, π'_B) over A and B, there is a unique arrow $\varphi : P' \to P$ such that $\pi_A.\varphi \leq \pi'_A$ and $\pi_B.\varphi \leq \pi'_B$ and $\overline{\varphi} = \overline{\pi'_A}\,\overline{\pi'_B}$:

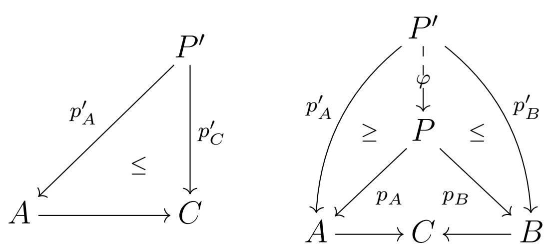


(b) Restricted pullbacks in a restriction category **X** : Given any cospan

$$A \longrightarrow C \longleftarrow B\ ,$$

a restricted pullback is cone consisting of an object P and total arrows $p_{A,B,C}$: $P \to A/B/C$ satisfying the following universal property:

For each lax cone (P', p'_A, p'_B, p'_C) over $A \longrightarrow B \longleftarrow C$, there is a unique φ : $P' \to P$ such that $\varphi \circ p \leq p'$ and $\overline{\varphi} = \overline{p'_A}\,\overline{p'_B}\,\overline{p'_C}$:



▲

Examples of Double Restriction Categories

Two natural examples of double restriction categories can be given, one of which requiring a restriction category with restricted pullbacks.

Example 5.1.10. Let **X** be a restriction category. Define a double restriction category with the following data:

- Objects: Objects of **X**.
- Vertical Arrows: Total arrows in **X**. Denote the identity arrow as 1_A. The restriction structure on the vertical arrows is that from **X** and is trivial: $\overline{f} = 1$

for all f.

- Horizontal Arrows: Arrows in $\mathbf{X}$. The horizontal restriction is also just the restriction from $\mathbf{X}$.

- Double cells: A typical double cell is a 2-cell

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ u \,\big\downarrow\!\bullet & \leq & \bullet\!\big\downarrow\, v \\ C & \xrightarrow[g]{} & D \end{array}$$

  - Vertical and horizontal composition of double cells are defined as usual (by composition along the boundaries) and preserve the partial order defining the double cells.
  - The vertical restriction of a double cell is defined by taking the vertical restriction of the vertical arrows comprising its horizontal domain and codomain. Horizontal restrictions of double cells are similarly defined. ▲

The following example is a natural generalization of the restriction category $\mathrm{Par}(\mathbf{X}, \mathcal{M})$ using the methods of the span construction [12].

Example 5.1.11. Let $\mathbf{X}$ be a restriction category with restricted pullbacks. Define a double restriction category with the following data:

- Objects: Objects of $\mathbf{X}$.

- Vertical Arrows: Total arrows in $\mathbf{X}$. Denote the identity arrow as 1_A. The restriction structure on the vertical arrows is that from $\mathbf{X}$ and is trivial: $\overline{f} = 1$ for all f.

- Horizontal Arrows: Arrows in $\mathrm{Par}(\mathbf{X}, \mathcal{M})$, where $\mathcal{M}$ is a system of monics stable under restricted pullbacks (cf. Example 2.0.2(b)). Denote the identity as $\mathrm{id}_A = (1_A, 1_A)$. Recall that the horizontal restriction structure can be given by $\overline{(i, f)} = (i, i)$ for each (i, f).

- Double cells: A typical double cell Γ_α is defined by an arrow $\alpha : D \to D'$ in $\mathbf{X}$ such that $i'\alpha \leq ui$ and $f'\alpha \leq vf$ in $\mathbf{X}$:

$$\Gamma_\alpha = \begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\!<} & D & \overset{f}{\longrightarrow} & Y \\ u\bullet\downarrow & \geq & \downarrow\alpha & \leq & \downarrow\bullet v \\ X' & \underset{i'}{\longleftarrow\!\!\!<} & D' & \underset{f'}{\longrightarrow} & Y' \end{array}$$

  - Vertical Composition: Given by composition in $\mathbf{X}$:

$$\begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\!<} & D & \overset{f}{\longrightarrow} & Y \\ u\bullet\downarrow & \geq & \downarrow\alpha & \leq & \downarrow\bullet v \\ X & \overset{i'}{\longleftarrow\!\!\!<} & D & \overset{f'}{\longrightarrow} & Y \\ u'\bullet\downarrow & \geq & \downarrow\alpha' & \leq & \downarrow\bullet v' \\ X'' & \underset{i''}{\longleftarrow\!\!\!<} & D'' & \underset{f''}{\longrightarrow} & Y'' \end{array} = \begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\!<} & D & \overset{f}{\longrightarrow} & Y \\ u'\bullet u\downarrow & \geq & \downarrow\alpha'\bullet\alpha & \leq & \downarrow v'\bullet v \\ X' & \underset{i''}{\longleftarrow\!\!\!<} & D' & \underset{f''}{\longrightarrow} & Y' \end{array}$$

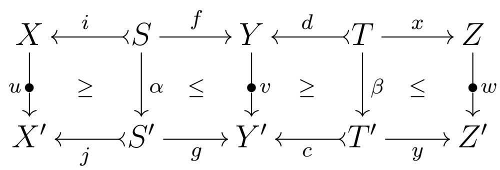
    The resulting composite is a double cell since $v'vf \leq v'f'\alpha \leq f''\alpha'\alpha$ and $i''\alpha'\alpha \leq u'i'\alpha \leq u'ui$.

  - Vertical Restriction: For each such Γ_α, define the vertical restriction $\widetilde{\Gamma_\alpha}$ to be

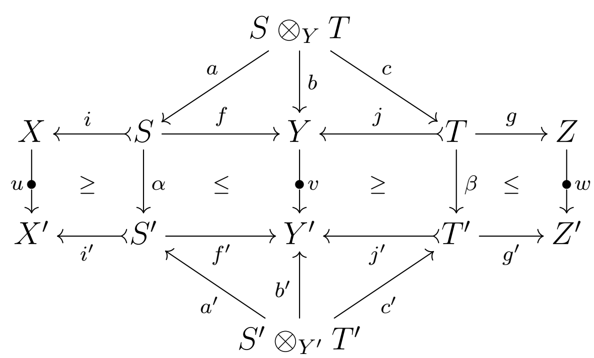
$$\widetilde{\Gamma_\alpha} = \begin{array}{ccccc} X & \overset{i}{\longleftarrow\!\!\!<} & D & \overset{f}{\longrightarrow} & Y \\ \overline{u}=1_X\,\Big\| & \geq & \downarrow\overline{\alpha} & \leq & \Big\|\,1_Y=\overline{v} \\ X & \underset{i}{\longleftarrow\!\!\!<} & D & \underset{f}{\longrightarrow} & Y \end{array}$$

Notice that $i\overline{i\overline{\alpha}} = i\,\overline{i}\,\overline{\alpha} = i\overline{\alpha}$ and thus $i\overline{\alpha} \leq i$. Similarly, $f\overline{\alpha} \leq f$ and the vertical restriction as defined is indeed a double cell. The conditions (R.1) – (R.4) making the vertical category of double cells and horizontal arrows a restriction category are easily verified.

- Horizontal Composition: We wish to horizontally compose the double cells Γ_α and Γ_β :

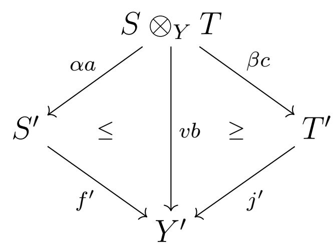


First take the restricted pullbacks:



Recall that this means, instead of strict equality in the cones, we have $jc \leq b$, $fa \leq b$, $f'a' \leq b'$ and $j'c' \leq b'$. Note that $j'\beta c \leq vjc \leq vb$ and

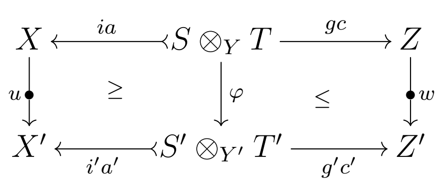
$f'\alpha a \leq vfa \leq vb$, so there is a lax cone



over

$$S' \xrightarrow{g} Y' \xleftarrow{c} T'$$

and there is therefore a unique $\varphi : S \otimes_Y T \to S' \otimes_{Y'} T'$ such that $a'\varphi \leq \alpha a$, $b'\varphi \leq vb$, $c'\varphi \leq \beta c$ and $\overline{\varphi} = \overline{\alpha a}\,\overline{\beta c}\,\overline{vb}$, fitting into a double cell

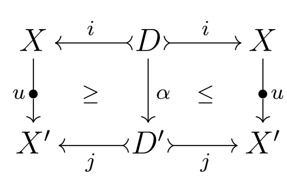


since

(i) $a, a' \in \mathcal{M}$ by stability of $\mathcal{M}$ and thus $ia, i'a' \in \mathcal{M}$,

(ii) $g'c'\varphi \leq g'\beta c \leq wgc$ and $i'a'\varphi \leq i'\alpha a \leq uia$ and thus the required inequalities hold.

– Horizontal Restriction: For each such Γ_α, define the horizontal restriction $\overline{\Gamma_\alpha}$ to be

$\overline{\Gamma_\alpha} =$ 

With Γ_α being a double cell, it follows immediately that $\overline{\Gamma_\alpha}$ is, too. The conditions (R.1) – (R.4) making the horizontal category of double cells and vertical arrows a restriction category are easily verified.

– It is quickly seen that the restriction structures commute:

$$
\begin{array}{ccccc}
X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\
{\scriptstyle u}\bullet\downarrow & \geq & \downarrow{\scriptstyle\alpha} & \leq & \downarrow\bullet{\scriptstyle v} \\
X' & \xleftarrow[i']{} & D' & \xrightarrow[f']{} & Y'
\end{array}
\xrightarrow{\widetilde{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\
\| & \geq & \downarrow{\scriptstyle\overline{\alpha}} & \leq & \| \\
X & \xleftarrow[i]{} & D & \xrightarrow[f]{} & Y
\end{array}
\xrightarrow{\overline{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{i} & D & \xrightarrow{i} & X \\
\| & \geq & \downarrow{\scriptstyle\overline{\alpha}} & \leq & \| \\
X & \xleftarrow[i]{} & D & \xrightarrow[i]{} & X
\end{array}
$$

$$
\begin{array}{ccccc}
X & \xleftarrow{i} & D & \xrightarrow{f} & Y \\
{\scriptstyle u}\bullet\downarrow & \geq & \downarrow{\scriptstyle\alpha} & \leq & \downarrow\bullet{\scriptstyle v} \\
X' & \xleftarrow[i']{} & D' & \xrightarrow[f']{} & Y'
\end{array}
\xrightarrow{\overline{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{i} & D & \xrightarrow{i} & X \\
{\scriptstyle u}\bullet\downarrow & \geq & \downarrow{\scriptstyle\alpha} & \leq & \downarrow\bullet{\scriptstyle u} \\
X' & \xleftarrow[j]{} & D' & \xrightarrow[j]{} & X'
\end{array}
\xrightarrow{\widetilde{(-)}}
\begin{array}{ccccc}
X & \xleftarrow{i} & D & \xrightarrow{i} & X \\
\| & \geq & \downarrow{\scriptstyle\overline{\alpha}} & \leq & \| \\
X & \xleftarrow[i]{} & D & \xrightarrow[i]{} & X
\end{array}
$$

- The middle-four interchange law for vertical and horizontal composition follows from the universal property of the restricted pullbacks defining composition. ▲

We end this section with a short note on the appropriate definition of double inverse semigroups. A double inverse semigroup, as defined in [24], is a set equipped with two inverse semigroup operations satisfying the middle-four interchange law. A set equipped with two such group operations is *improper* – the operations coincide – and commutative; this is known as the Eckmann-Hilton argument [14]. This result extends to double inverse semigroups [13] and thus an element-based definition of a double inverse semigroup is not useful. Brown [5] redefined a double group as a single-object double groupoid. The benefit of this definition is that we encode two group operations interacting via the interchange law on the double cells, but the underlying sets on which the group operations are defined need not be equal. Motivated by this, we make the following definition:

Definition 5.1.12. A *double inverse semigroup* is a single-object double inverse semicategory. ♢

One may immediately wonder if Definition 5.1.12 has any interesting applications.

Question. Which results in inverse semigroup theory can be appropriately "doubled" to this context of double inverse semigroups?

A *crossed module* is an action of a group G on a group H together with a group homomorphism

$$\varphi : H \to G$$

satisfying $\varphi(g \cdot h) = g\varphi(h)g^{-1}$ and $\varphi(h_1) \cdot h_2 = h_1 h_2 h_1^{-1}$. In some sense, crossed modules are two-dimensional groups (two group structures behaving nicely with each other) and were first used to relate topological structure: if (X, A, x) is a pointed pair of topological spaces, then the boundary map

$$\partial : \pi_2(X, A, x) \to \pi_1(A, x)$$

is a crossed module with the induced functor

$$\Pi : \textbf{PairsPointedSpaces} \to \textbf{CrossedModules}$$

satisfying the van Kampen Theorem [4].

Replacing these groups with inverse semigroups, a very open ended question we would like to see answered in the long term:

Question. Is there a generalized notion of homotopy so that the formulation of the fundamental *inverse category* (versus the fundamental groupoid) makes sense? Then, do crossed modules translate to the language of inverse semigroup actions? Can these be seen as a restriction module of an inverse category acting somehow on a space?

As this structure would be in some sense a two-dimensional inverse semigroup, it would then be interesting to know if these can be reformulated as double inverse semigroups (or double inverse categories).

5.2 Restriction Bicategories

This section contains a motivating example for the definition of a restriction bicategory, a bicategory with a functorial restriction structure on its 1-cells satisfying (R1) through (R4) up to invertible 2-cell.

Composing Restriction Bimodules

We compose restriction bimodules (cf. Definition 3.2.1) in the usual way:

Definition 5.2.1. If $\mathbf{X} \overset{\varphi}{\longmapsto} \mathbf{Y} \overset{\psi}{\longmapsto} \mathbf{Z}$ are restriction bimodules then $\psi \otimes \varphi : \mathbf{X} \longmapsto \mathbf{Z}$ is defined by the coequalizer diagram

$$\coprod_{y_1, y_2 \in \mathbf{Y}_0} \psi(z,y) \times \mathbf{Y}(y_1, y_2) \times \varphi(y_2, x) \underset{1 \times \rho^{\varphi}}{\overset{\lambda^{\psi} \times 1}{\rightrightarrows}} \coprod_{y \in \mathbf{Y}_0} \psi(z,y) \times \varphi(y,x) \longrightarrow (\psi \otimes \varphi)(z,x)$$

with left and right action defined by $(\beta \otimes \alpha) \cdot f = \beta \otimes (\alpha \cdot f)$ and $g \cdot (\beta \otimes \alpha) = (g \cdot \beta) \otimes \alpha$. $\diamondsuit$

Without loss of generality, we consider an arbitrary element of $\psi \otimes \varphi$ as having the form $\beta \otimes \alpha$ with $\alpha \in \varphi(y,x)$ and $\beta \in \psi(z,y)$.

Proposition 5.2.2. *If* $\mathbf{X} \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\;\mapsto} \mathbf{Y} \overset{\psi}{\longrightarrow\!\!\!\!\!\!\!\!\;\mapsto} \mathbf{Z}$ *are restriction bimodules then* $\psi \otimes \varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\!\!\;\mapsto \mathbf{Z}$ *is a restriction bimodule with* $\overline{\beta \otimes \alpha} = \overline{\overline{\beta} \cdot \alpha}$.

Proof. To prove (RMod.0), suppose that $\beta \otimes \alpha \in (\psi \otimes \varphi)(z, x)$ with $\alpha \in \varphi(y', x)$ and $\beta \in \psi(z, y')$. Then $\overline{\beta} \cdot \alpha \in \varphi(y', x)$ and thus

$$\overline{\beta \otimes \alpha} = \overline{\overline{\beta} \cdot \alpha}$$

is a restriction idempotent in $\mathbf{X}$ by (RMod.0) of φ.

To prove (RMod.1), suppose that $\beta \otimes \alpha \in (\psi \otimes \varphi)(z, x)$. Then

$$\begin{aligned}
(\beta \otimes \alpha) \cdot \overline{\beta \otimes \alpha} &= (\beta \otimes \alpha) \cdot \overline{\overline{\beta} \cdot \alpha} \\
&= \beta \otimes (\alpha \cdot \overline{\overline{\beta} \cdot \alpha}) \\
&= \beta \otimes (\overline{\overline{\beta}} \cdot \alpha) \\
&= (\beta \cdot \overline{\beta}) \otimes \alpha \\
&= \beta \otimes \alpha.
\end{aligned}$$

To prove (RMod.3), suppose that $\beta \otimes \alpha \in (\psi \otimes \varphi)(z, x)$ and $\gamma \otimes \delta \in (\psi \otimes \varphi)(z', x)$. Then

$$\begin{aligned}
\overline{(\beta \otimes \alpha) \cdot \overline{\gamma \otimes \delta}} &= \overline{\beta \otimes (\alpha \cdot \overline{\overline{\gamma} \cdot \delta})} \\
&= \overline{(\overline{\beta} \cdot \alpha) \cdot \overline{\overline{\gamma} \cdot \delta}} \\
&= \overline{\overline{\beta} \cdot \alpha} \circ \overline{\overline{\gamma} \cdot \delta} \\
&= \overline{\beta \otimes \alpha} \circ \overline{\gamma \otimes \delta}.
\end{aligned}$$

To prove (RMod.4), suppose that $\beta \otimes \alpha \in (\psi \otimes \varphi)(z, x)$ and $\gamma \otimes \delta \in (\psi \otimes \varphi)(z', x)$. Then

$$
\begin{aligned}
f \circ \overline{(\beta \otimes \alpha) \cdot f} &= f \circ \overline{\beta \otimes (\alpha \cdot f)} \\
&= f \circ \overline{\overline{\beta} \cdot (\alpha \cdot f)} \\
&= \overline{\overline{\beta} \cdot \alpha} \circ f \\
&= \overline{\beta \otimes \alpha} \circ f
\end{aligned}
$$

and

$$
\begin{aligned}
(\beta \otimes \alpha) \cdot \overline{g \cdot (\beta \otimes \alpha)} &= (\beta \otimes \alpha) \cdot \overline{(g \cdot \beta) \otimes \alpha} \\
&= \beta \otimes (\alpha \cdot \overline{(g \cdot \beta) \cdot \alpha}) \\
&= \beta \otimes (\overline{\overline{g \cdot \beta}} \cdot \alpha) \\
&= \beta \otimes (\overline{g \cdot \beta} \cdot \alpha) \\
&= (\beta \cdot \overline{g \cdot \beta}) \otimes \alpha \\
&= (\overline{g} \cdot \beta) \otimes \alpha \\
&= \overline{g} \cdot (\beta \otimes \alpha).
\end{aligned}
$$
□

Restriction Bicategory of Restriction Bimodules

With composition of restriction bimodules associative only up to invertible 2-cell, we only require that conditions (R1) through (R4) would hold up to (coherent) invertible 2-cells.

Definition 5.2.3. Let **X** be a bicategory. A restriction structure on **X** is a famiy of functors

$$\overline{(-)} : \mathbf{X}(A, B) \to \mathbf{X}(A, A)$$

indexed by the 0-cells of $\mathbf{X}$ together with invertible 2-cells

(i) $\rho_1 : f\overline{f} \cong f$

(ii) $\rho_2 : \overline{f}\,\overline{g} \cong \overline{g}\,\overline{f}$

(iii) $\rho_3 : \overline{g\overline{f}} \cong \overline{g}\overline{f}$

(iv) $\rho_4 : \overline{g}f \cong f\,\overline{gf}$

The assignment of restriction operators being functorial, in particular, implies that if $\alpha : f \cong g$ then $\overline{\alpha} : \overline{f} \cong \overline{g}$.

In addition to the usual pentagonal and unit coherence diagrams, the following diagrams must also commute for any isomorphism $\alpha : f \cong g$:

$$\begin{array}{ccc} f & \xrightarrow{\rho_1} & f\,\overline{f} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\overline{\alpha}} \\ g & \xrightarrow[\rho_1]{} & g\,\overline{g} \end{array} \qquad \begin{array}{ccccc} \overline{g}\,\overline{g} & \xleftarrow{\overline{\alpha}\,\overline{g}} & \overline{f}\,\overline{g} & \xrightarrow{\overline{f}\,\overline{\alpha}^{-1}} & \overline{f}\,\overline{f} \\ & {\scriptstyle\overline{g}\,\overline{\alpha}^{-1}}\searrow & \downarrow{\scriptstyle\rho_2} & \swarrow{\scriptstyle\overline{\alpha}\,\overline{f}} & \\ & & \overline{g}\,\overline{f} & & \end{array}$$

$$\begin{array}{ccc} \overline{g\,\overline{f}} & \xrightarrow{\rho_3} & \overline{g}\,\overline{f} \\ {\scriptstyle\overline{\alpha^{-1}\,\overline{\alpha}}}\downarrow & & \downarrow{\scriptstyle\overline{\alpha}^{-1}\overline{\alpha}} \\ \overline{f\,\overline{g}} & \xrightarrow[\rho_3]{} & \overline{g}\,\overline{f} \end{array} \qquad \begin{array}{ccc} \overline{f'}\,f & \xrightarrow{\rho_4} & f\,\overline{f'f} \\ {\scriptstyle\overline{\alpha'}\,\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\,\overline{\alpha'\alpha}} \\ \overline{g'}\,g & \xrightarrow[\rho_4]{} & g\,\overline{g'g} \end{array}$$

◊

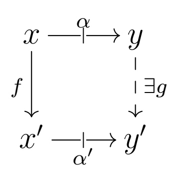
Question. Are these diagrams sufficient to show coherence in a restriction bicategory, or are more conditions required?

That two isomorphic 1-cells must have isomorphic restrictions allows us to prove that all of the usual conditions for a restriction category are satisfied, with equality being up to distinguished invertible 2-cell. E.g., that $\overline{f}\,\overline{f} \cong \overline{f}$ or that $\overline{f}\,\overline{gf} \cong \overline{gf}$:

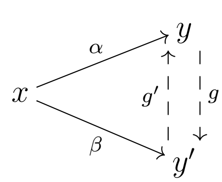
$$\overline{f}\,\overline{gf} \xrightarrow[\rho_2]{\cong} \overline{gf}\,\overline{f} \xrightarrow[\rho_3^{-1}]{\cong} \overline{gf\overline{f}} \xrightarrow[\overline{1_g\,\rho_1}]{\cong} \overline{gf}$$

For the next example, one needs a new condition: if $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\!\!\!\!\mapsto \mathbf{Y}$ is a restriction bimodule, then we require that

(*) For all $f : x \to x'$ in $\mathbf{X}$, $\alpha \in \varphi(y,x)$ and $\alpha' \in \varphi(y',x')$, there exists $g : y \to y'$ such that $g \cdot \alpha = \alpha' \cdot f$. That is, the following diagram can always be completed to be commutative:



Such a condition is quite strong in the case of restriction categories but is quite natural when restricted to inverse categories; given the following diagram, there exist g, g' with $(gg'g) \cdot \alpha = g \cdot \alpha$ and $(g'gg') \cdot \beta = g' \cdot \beta$:



The necessity of this condition will become clear when proving that ρ_4 exists for the bicategory of restriction bimodules.

Definition 5.2.4. A morphism $F : \varphi \Rightarrow \psi : \mathbf{X} \relbar\!\mapsto \mathbf{Y}$ of left-$\mathbf{Y}$ right-$\mathbf{X}$ restriction bimodules φ and ψ is a family of functions $F_{y,x} : \varphi(y,x) \to \psi(y,x)$ such that

- F is equivariant with respect to the action of $\mathbf{X}$ and $\mathbf{Y}$ on φ and ψ :

$$F(\alpha \cdot f) = F(\alpha) \cdot f \text{ and } F(g \cdot \alpha) = g \cdot F(\alpha).$$

- F is compatible with the restriction structure in the sense that, for all α, $\overline{F\alpha} = \overline{\alpha}$. This condition on a morphism of restriction bimodules is analogous to requiring that restriction functors between restriction categories preserve restriction idempotents. $\diamondsuit$

Example 5.2.5. This example defines the restriction bicategory rModule(**rCat**) of restriction bimodules. If $\varphi : \mathbf{X} \relbar\!\mapsto \mathbf{Y}$ is a restriction bimodule satisfying condition (*), define a new module $\overline{\varphi} : \mathbf{X} \relbar\!\mapsto \mathbf{X}$

$$\overline{\varphi}(x', x) = \{f \circ \overline{\alpha} : \alpha \in \varphi(y,x), f : x \to x'\}$$

Actions are given by composition. Conditions for being a restriction bicategory:

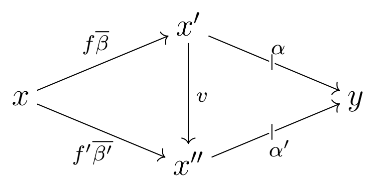
(i) $\varphi \otimes \overline{\varphi} \cong \varphi$

Define a map by

$$\rho_1 : \alpha \otimes (f\overline{\beta}) \mapsto \alpha \cdot (f\overline{\beta}).$$

Note that this map is immediately left and right equivariant and therefore a bimodule morphism.

We show now that it is well defined. Suppose that there is a map $v : x' \to x''$ in **X** making the following diagram commute:



Then

$$
\begin{aligned}
\alpha \cdot (f\overline{\beta}) &= (\alpha' \cdot v) \cdot (f\overline{\beta}) \\
&= \alpha' \cdot (v(f\overline{\beta})) \\
&= \alpha' \cdot (f'\overline{\beta'})
\end{aligned}
$$

and this map is well defined. There is also a map $\varphi \to \varphi \otimes \overline{\varphi}$ defined by $\alpha \mapsto \alpha \cdot \overline{\alpha}$ which is similarly a well define module morphism. This map is the inverse of ρ_1, making ρ_1 an isomorphism of bimodules:

$$\alpha \mapsto \alpha \otimes \overline{\alpha} \mapsto \alpha \cdot \overline{\alpha} = \alpha$$

and

$$\begin{aligned}
\alpha \cdot (f\overline{\beta}) &\mapsto [\alpha \cdot (f\overline{\beta})] \otimes \overline{\alpha \cdot (f\overline{\beta})} \\
&= \alpha \otimes f\overline{\beta}\,\overline{\alpha \cdot (f\overline{\beta})} \\
&= \alpha \otimes \overline{\alpha}(f\overline{\beta}) \\
&= \alpha \cdot \overline{\alpha} \otimes f\overline{\beta} \\
&= \alpha \otimes f\overline{\beta} \\
&\mapsto \alpha \cdot (f\overline{\beta})
\end{aligned}$$

(ii) $\overline{\varphi} \otimes \overline{\psi} \cong \overline{\psi} \otimes \overline{\varphi}$

Define a map by

$$\rho_2 : (g\overline{\beta}) \otimes (f\overline{\alpha}) \mapsto (gf\overline{\alpha}) \otimes \overline{\beta \cdot f}$$

This map is clearly left equivariant. We also quickly see that it is right equivariant:

$$\begin{aligned}
\rho_2([g\overline{\beta} \otimes f\overline{\alpha}] \cdot A) &= \rho_2(g\overline{\beta} \otimes f\overline{\alpha}A) \\
&= \rho_2(g\overline{\beta} \otimes fA\,\overline{\alpha \cdot A}) \\
&= gfA\,\overline{\alpha \cdot A} \otimes \overline{\beta \cdot (fA)} \\
&= gf\overline{\alpha}A \otimes \overline{\beta \cdot (fA)} \\
&= gf\overline{\alpha} \otimes A\overline{(\beta \cdot f)A} \\
&= gf\overline{\alpha} \otimes \overline{(\beta \cdot f)}A \\
&= \rho_2([g\overline{\beta} \otimes f\overline{\alpha}]) \cdot A
\end{aligned}$$

We show now that it is well defined. Suppose that there is a map $v : x'' \to x'''$ in $\mathbf{X}$ such that $vf\overline{\alpha} = f'\overline{\alpha'}$ and $g'\overline{\beta'}v = g\overline{\beta}$. Then

$$\begin{aligned}
g'f'\overline{\alpha'} \otimes \overline{\beta' \cdot f'} &= g'f'\overline{\alpha'}\,\overline{\beta' \cdot f'} \otimes \overline{\beta' \cdot f'} \\
&= g'f'\overline{\beta' \cdot f'}\,\overline{\alpha'} \otimes \overline{\beta' \cdot f'} \\
&= g'\overline{\beta'}f'\,\overline{\alpha'} \otimes \overline{\beta' \cdot f'} \\
&= g'\overline{\beta'}vf\,\overline{\alpha} \otimes \overline{\beta' \cdot f'} \\
&= g\overline{\beta}f\,\overline{\alpha} \otimes \overline{\beta' \cdot f'} \\
&= gf\overline{\beta \cdot f}\,\overline{\alpha} \otimes \overline{\beta' \cdot f'} \\
&= gf\overline{\beta \cdot f}\,\overline{\beta' \cdot f'}\,\overline{\alpha} \otimes \overline{\beta \cdot f} \\
&= g\overline{\beta}f\,\overline{\beta' \cdot f'}\,\overline{\alpha} \otimes \overline{\beta \cdot f} \\
&= g'\overline{\beta'}vf\,\overline{\alpha}\,\overline{\beta' \cdot f'} \otimes \overline{\beta \cdot f} \\
&= g'\overline{\beta'}f'\,\overline{\alpha'}\,\overline{\beta' \cdot f'} \otimes \overline{\beta \cdot f} \\
&= g'\overline{\beta'}f'\,\overline{\beta' \cdot f'}\,\overline{\alpha'} \otimes \overline{\beta \cdot f} \otimes \overline{\beta \cdot f} \\
&= g'\overline{\beta'}\,\overline{\beta'}f'\,\overline{\alpha'} \otimes \overline{\beta \cdot f} \\
&= g'\overline{\beta'}\,vf\,\overline{\alpha} \otimes \overline{\beta \cdot f}\,\overline{\beta \cdot f} \\
&= g\overline{\beta}\,f\,\overline{\alpha} \otimes \overline{\beta \cdot f} \\
&= gf\overline{\beta \cdot f}\,\overline{\alpha} \otimes \overline{\beta \cdot f} \\
&= gf\overline{\alpha} \otimes \overline{\beta \cdot f}
\end{aligned}$$

and this map is well defined. Injective: consider

$$(g\overline{\beta}) \otimes (f\overline{\alpha}) \text{ and } (g'\overline{\beta'}) \otimes (f'\overline{\alpha'})$$

and assume that

$$(gf\overline{\alpha}) \otimes \overline{\beta \cdot f} = (g'f'\overline{\alpha'}) \otimes \overline{\beta' \cdot f'}.$$

Then this map is injective since

$$\begin{aligned}
(gf\overline{\alpha}) \otimes \overline{\beta \cdot f} &= gf\overline{gf}\,\overline{\beta \cdot f}\,\overline{\alpha} \otimes 1 \\
&= gf\overline{\beta \cdot f}\,\overline{\alpha} \otimes 1 \\
&= g\overline{\beta}f\overline{\alpha} \otimes 1 \\
&= g\overline{\beta} \otimes f\overline{\alpha}.
\end{aligned}$$

Surjective: given $(f\overline{\alpha}) \otimes (g\overline{\beta}) \in \overline{\varphi} \otimes \overline{\psi}(x', x)$, we have

$$(fg\overline{\beta}) \otimes \overline{\alpha \cdot g} \in \overline{\psi} \otimes \overline{\varphi}$$

with

$$\begin{aligned}
(fg\overline{\beta}) \otimes \overline{\alpha \cdot g} \mapsto (fg\,\overline{\alpha \cdot g}\,\overline{fg}\,\overline{\beta}\,\overline{fg}) \otimes 1 \\
&= (fg\,\overline{\alpha \cdot g}\,\overline{\beta}) \otimes 1 \\
&= (f\,\overline{\alpha}\,g\,\overline{\beta}) \otimes 1 \\
&= (f\overline{\alpha}) \otimes (g\overline{\beta})
\end{aligned}$$

(iii) $\overline{\varphi \otimes \overline{\psi}} \cong \overline{\varphi} \otimes \overline{\psi}$

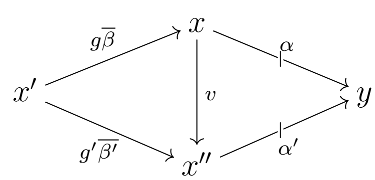
Define a map by

$$\rho_3 : f\,\overline{\alpha \otimes (g\overline{\beta})} \mapsto (f\,\overline{\alpha \cdot g}) \otimes \overline{\beta}$$

This map is clearly left equivariant. We also quickly see that it is right equivariant:

$$\begin{aligned}
\rho_3(f\overline{\alpha \otimes g\overline{\beta}}A) &= \rho_3((fA)\overline{\alpha \otimes g\overline{\beta}A}) \\
&= \rho_3((fA)\overline{\alpha \otimes gA\overline{\beta \cdot A}}) \\
&= fA\,\overline{\alpha \cdot (gA)} \otimes \overline{\beta \cdot A} \\
&= fA\,\overline{(\alpha \cdot g) \cdot A} \otimes \overline{\beta \cdot A} \\
&= f\,\overline{(\alpha \cdot g)}A \otimes \overline{\beta \cdot A} \\
&= f\,\overline{(\alpha \cdot g)} \otimes A\overline{\beta \cdot A} \\
&= f\,\overline{(\alpha \cdot g)} \otimes \overline{\beta}A \\
&= \rho_3(f\overline{\alpha \otimes g\overline{\beta}}) \cdot A
\end{aligned}$$

We show now that it is well defined. Suppose that there is a map $v : x \to x''$ in $\mathbf{X}$ making the following diagram commute:



Then

$$
\begin{aligned}
(f\,\overline{\alpha \cdot g}) \otimes \overline{\beta} &= f\overline{(\alpha' \cdot v) \cdot g} \otimes \overline{\beta} \\
&= f\overline{(\alpha' \cdot v) \cdot g} \otimes \overline{(\alpha' \cdot v) \cdot g}\,\overline{\beta} \\
&= f\overline{(\alpha' \cdot v) \cdot g} \otimes \overline{(\alpha' \cdot v) \cdot g \cdot \overline{\beta}} \\
&= f\overline{\alpha' \cdot (vg)} \otimes \overline{\alpha' \cdot (g' \cdot \overline{\beta'})} \\
&= f\overline{\alpha' \cdot (vg)} \otimes \overline{\alpha' \cdot g'}\,\overline{\beta'} \\
&= f\overline{\alpha' \cdot (vg)}\,\overline{\alpha' \cdot g'}\,\overline{\beta'} \otimes \overline{\beta'} \\
&= f\overline{(\alpha' \cdot (vg)) \cdot (\overline{\alpha' \cdot g'}\,\overline{\beta'})} \otimes \overline{\beta'} \\
&= f\overline{(\alpha' \cdot (vg\overline{\beta})) \cdot \overline{\alpha' \cdot g'})} \otimes \overline{\beta'} \\
&= f\overline{(\alpha' \cdot (g'\overline{\beta'})) \cdot \overline{\alpha' \cdot g'})} \otimes \overline{\beta'} \\
&= f\overline{\alpha' \cdot (g'\overline{\beta'}\,\overline{\alpha' \cdot g'})} \otimes \overline{\beta'} \\
&= f\overline{\alpha' \cdot (g'\overline{\alpha' \cdot g'}\,\overline{\beta'})} \otimes \overline{\beta'} \\
&= f\overline{((\alpha' \cdot g') \cdot \overline{\alpha' \cdot g'})\,\overline{\beta'})} \otimes \overline{\beta'} \\
&= f\overline{\alpha' \cdot g'}\,\overline{\beta'} \otimes \overline{\beta'} \\
&= f\overline{\alpha' \cdot g'} \otimes \overline{\beta'}
\end{aligned}
$$

and this map is well defined.

Injective: consider

$$
f\,\overline{\alpha \otimes (g\overline{\beta})}, f'\,\overline{\alpha' \otimes (g'\overline{\beta'})} \in \overline{\varphi \otimes \overline{\psi}}(x', x)
$$

and assume that

$$(f\,\overline{\alpha\cdot g})\otimes\overline{\beta}=(f'\,\overline{\alpha'\cdot g'})\otimes\overline{\beta'}$$

Then this map is injective since

$$\begin{aligned}(f\,\overline{\alpha\cdot g})\otimes\overline{\beta}&=f\,\overline{\alpha\cdot g}\,\overline{\beta}\otimes 1\\&=f\,\overline{\overline{\alpha}\,g}\,\overline{\beta}\otimes 1\\&=f\,\overline{\overline{\alpha}\,g\,\overline{\beta}}\otimes 1\\&=f\,\overline{\alpha\otimes(g\overline{\beta})}\otimes 1\end{aligned}$$

Surjective: given $(f\overline{\alpha})\otimes g\overline{\beta}\in\overline{\varphi}\otimes\overline{\psi}(x',x)$, we have

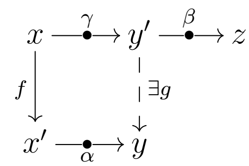
$$fg\,\overline{\alpha\otimes(g\overline{\beta})}\in\overline{\varphi\otimes\overline{\psi}}(x',x)$$

with

$$\begin{aligned}fg\,\overline{\alpha\otimes(g\overline{\beta})}&\mapsto fg\,\overline{\alpha\cdot g}\otimes\overline{\beta}\\&=f\overline{\alpha}g\otimes\overline{\beta}\\&=f\overline{\alpha}\otimes g\overline{\beta}.\end{aligned}$$

(iv) $\overline{\psi}\otimes\varphi\cong\varphi\otimes\overline{\psi\otimes\varphi}$

Recall that an element in $\varphi \otimes \overline{\psi \otimes \varphi}(y, x)$ is of the shape $\alpha \otimes f\overline{\beta \otimes \gamma}$:



Where the map g is guaranteed by condition (*). Define a map by

$$\rho_4 : f\overline{\beta} \otimes \alpha \mapsto ((f\overline{\beta}) \cdot \alpha) \otimes \overline{\beta \otimes \alpha}$$

with inverse

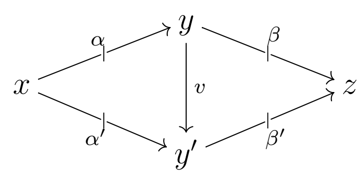
$$\rho_4^{-1} : \alpha \otimes f\,\overline{\beta \otimes \gamma} \mapsto (g\overline{\beta}) \otimes \gamma$$

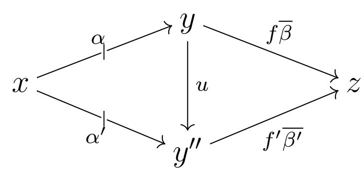
This map is clearly left equivariant and is easily seen to be right equivariant since

$$\begin{aligned}
\rho_4(f\overline{\beta} \otimes (\alpha \cdot A)) &= (f\overline{\beta}) \cdot (\alpha \cdot A) \otimes \overline{\beta \otimes (\alpha \cdot A)} \\
&= ((f\overline{\beta}) \cdot \alpha) \cdot A \otimes \overline{(\beta \otimes \alpha) \cdot A} \\
&= ((f\overline{\beta}) \cdot \alpha) \otimes A\overline{(\beta \otimes \alpha) \cdot A} \\
&= ((f\overline{\beta}) \cdot \alpha) \otimes \overline{(\beta \otimes \alpha)}A \\
&= \rho_4(f\overline{\beta} \otimes \alpha) \cdot A
\end{aligned}$$

We show now that it is well defined. Suppose that there are maps $v : y \to y'$

$u : y \to y''$ in $\mathbf{Y}$ making the following diagrams commute:





Then

$$
\begin{aligned}
(f\overline{\beta}) \cdot \alpha \otimes \overline{\beta \otimes \alpha} &= (f'\overline{\beta'}u) \cdot \alpha \otimes \overline{\beta' \cdot v \otimes \alpha} \\
&= (f'\overline{\beta'}) \cdot (u \cdot \alpha) \otimes \overline{\beta' \otimes v \cdot \alpha} \\
&= (f'\overline{\beta'}) \cdot \alpha' \otimes \overline{\beta' \otimes \alpha'}
\end{aligned}
$$

and this map is well defined.

Injective: follows from

$$
\begin{aligned}
\alpha \otimes f\,\overline{\beta \otimes \gamma} &= (\alpha \cdot f) \otimes \overline{\beta \otimes \gamma} \\
&= (g \cdot \gamma) \otimes \overline{\beta \otimes \gamma} \\
&= (g \cdot \gamma)\overline{\beta \otimes \gamma} \otimes 1 \\
&= (g\overline{\beta}) \cdot \gamma \otimes 1 \\
&= g\overline{\beta} \otimes \gamma.
\end{aligned}
$$

Surjectivity is immediate:

$$(g \cdot \gamma) \otimes \overline{\beta \otimes \gamma} \mapsto g\overline{\beta} \otimes \gamma.$$

Finally, if $F : \varphi \cong \psi$, we define

$$\overline{F} : \overline{\varphi} \to \overline{\psi}$$

by

$$\overline{F}(f \circ \overline{\alpha}) = f \circ \overline{F\alpha}.$$

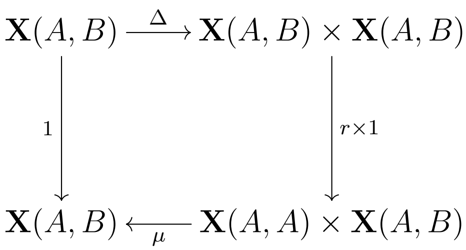
Since $\overline{F\alpha} = \overline{\alpha}$, this map is the identity and the four restriction coherence conditions are easily satisfied. ▲

5.3 Restriction Enriched Categories

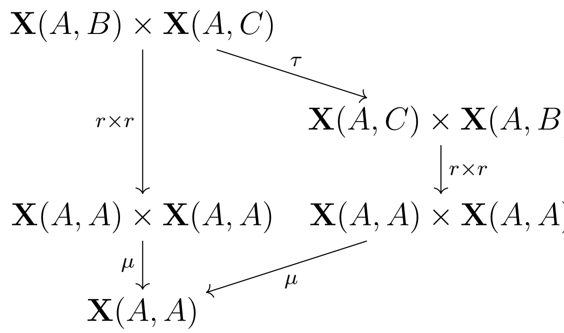
Definition 5.3.1. If $\mathcal{V}$ is a Cartesian monoidal category, then a restriction $\mathcal{V}$-category $\mathbf{X}$ is a $\mathcal{V}$-category equipped with, for each pair $A, B \in \mathbf{X}_0$ of objects, a $\mathcal{V}$-morphism

$$r_{A,B} : \mathbf{X}(A, B) \to \mathbf{X}(A, A)$$

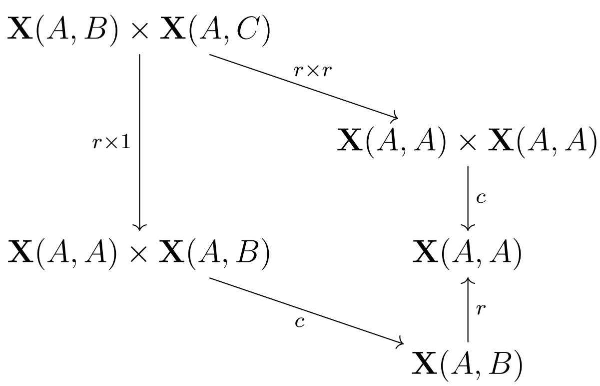
such that the following diagrams commute for all $A, B \in \mathbf{X}_0$:


X(A, B)
Δ
X(A, B) × X(A, B)
1
r×1
X(A, B)
μ
X(A, A) × X(A, B)

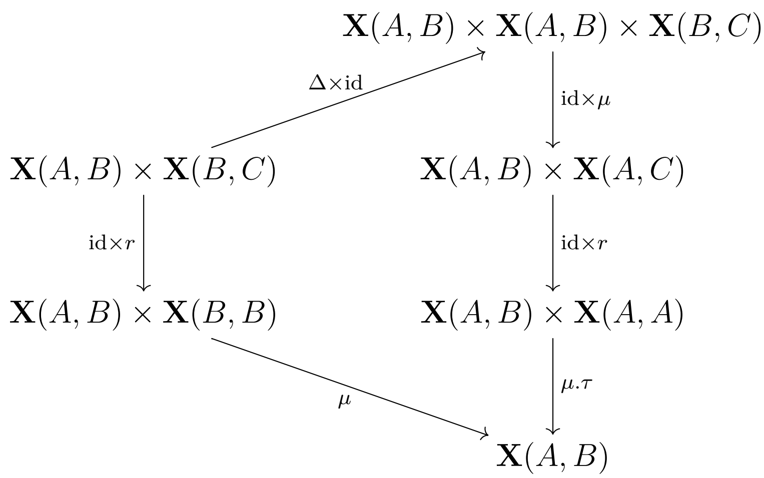
(R.1)


X(A, B) × X(A, C)
τ
r×r
X(A, C) × X(A, B)
r×r
X(A, A) × X(A, A)
X(A, A) × X(A, A)
μ
μ
X(A, A)

(R.2)


X(A, B) × X(A, C)
r×r
r×1
X(A, A) × X(A, A)
c
X(A, A) × X(A, B)
X(A, A)
r
c
X(A, B)

(R.3)



(R.4)

◇

Proposition 3.1.3 shows that restriction monads in **Set**-Mat are precisely small – that is, **Set**-enriched – restriction categories. This correspondence can be extended to restriction categories enriched in any suitable category $\mathcal{V}$.

Theorem 5.3.2. *If $\mathcal{V}$ is a Cartesian monoidal category, then restriction $\mathcal{V}$-categories correspond to restriction monads in $\mathcal{V}$-Mat.*

Proof. It is well known that a monad $T : \mathbf{X}_0 \to \mathbf{X}_0$ in $\mathcal{V}$-Mat (the bicategory whose 0-cells are sets and whose 1-cells are $\mathcal{V}$-valued matrices) – a matrix $T : \mathbf{X}_0 \times \mathbf{X}_0 \to \mathcal{V}$ – corresponds to a $\mathcal{V}$-category $\mathbf{X}$ whose hom-objects are defined by $\mathbf{X}(A, B) = T(B, A)$, whose multiplication is given by the $\mathcal{V}$-morphism $\mu : T^2 \Rightarrow T$ and whose unit is given by the $\mathcal{V}$-morphism $\eta : \mathrm{id}_{\mathbf{X}_0} \Rightarrow T$. As before, the hom-set $\mathcal{V}\text{-Mat}(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A})$ can be identified with the $\mathcal{V}$-object $T(B,A) = \mathbf{X}(A,B)$. The restriction $\mathcal{V}$-morphisms

$$\rho_{A,B} : \mathcal{V}\text{-Mat}(\{*\}, \mathbf{X}_0)(\vec{B}, T\vec{A}) \to \mathcal{V}\text{-Mat}(\{*\}, \mathbf{X}_0)(\vec{A}, T\vec{A})$$

are then in correspondence with the $\mathcal{V}$-morphisms

$$r_{A,B} : \mathbf{X}(A,B) \to \mathbf{X}(A,A),$$

which, by virtue of the $\mathcal{V}$-morphisms Δ and τ afforded by $\mathcal{V}$ being Cartesian monoidal, satisfy each of (R1) through (R4). □

Example 5.3.3. **Cat**-categories $\mathcal{B}$ are strict bicategories (or, 2-categories). A restriction structure on such a $\mathcal{B}$, then, is a family of functors $r_{A,B} : \mathcal{B}(A,B) \to \mathcal{B}(A,A)$ satisfying (R1) through (R4). In particular, the existence of an invertible $\alpha : f \Rightarrow g$ automatically gives an isomorphism $r(\alpha) : rf \cong rg$; a restriction **Cat**-category is a strict restriction bicategory. ▲

5.4 Supported Range Modules

In this section we introduce the double categories $\mathbb{R}$Module(**rCat**) and $s\mathbb{R}$Module(**rCat**), using blackboard bold to differentiate these from their horizontal bicategories rModule(**rCat**) and srModule(**rCat**).

The Double Category $\mathbb{R}$Module(rCat)

Definition 5.4.1. Let $\mathbf{C}$ be a restriction category. Define the double category $\mathbb{R}$Module(**rCat**) as follows:

- $\text{Obj}(\mathbb{R}\text{Module}(\mathbf{rCat})) = \mathbf{rCat}_0$.
- $\text{Ver}(\mathbb{R}\text{Module}(\mathbf{rCat})) = \mathbf{rCat}_1$.
- $\text{Hor}(\mathbb{R}\text{Module}(\mathbf{rCat})) = \text{rModule}(\mathbf{rCat})$.
- $\text{Dbl}(\mathbb{R}\text{Module}(\mathbf{rCat}))$: a double cell M in $\mathbb{R}\text{Module}(\mathbf{rCat})$

$$\begin{array}{ccc} \mathbf{X} & \overset{\varphi}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\,} & \mathbf{X}' \\ {\scriptstyle F}\downarrow & M & \downarrow{\scriptstyle F'} \\ \mathbf{Y} & \underset{\psi}{\longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\;\,} & \mathbf{Y}' \end{array}$$

  is a bimodule morphism $M : \varphi \to \psi$ which is equivariant with respect to F and F' : for all appropriate $f \in \mathbf{X}$ and $g \in \mathbf{Y}$, $M(\alpha \cdot f) = (M\alpha) \cdot (Ff)$ and $M(g \cdot \alpha) = (F'g) \cdot (M\alpha)$. $\diamondsuit$

Finally, we can organize the data structures defined in this thesis into two double categories (should we take the restriction monads in a double category so that the constructions are functorial):

	$\mathbb{R}$Module(**rCat**)	$\mathbb{R}$Mod(Span(**Set**))
Objects	Rest. Cats.	Rest. Monads in Span(**Set**)
Vertical Arrows	Rest. Functors	Monad Morphisms
Horizontal Arrows	Rest. Modules	Algebras
Double Cells	Equivariant Maps	Equivariant Maps

$$\begin{array}{ccc} \mathbf{X} & \overset{M}{-\!\!\!\mapsto} & \mathbf{X}' \\ F\downarrow & \alpha & \downarrow F' \\ \mathbf{Y} & \underset{M'}{-\!\!\!\mapsto} & \mathbf{Y}' \end{array} \qquad\qquad \begin{array}{ccc} T & \overset{A}{-\!\!\!\mapsto} & T' \\ F\downarrow & \alpha & \downarrow F' \\ N & \underset{B}{-\!\!\!\mapsto} & N' \end{array}$$

$\mathbb{R}$Module(**rCat**)) $\qquad$ $\mathbb{R}$Mod(Span(**Set**)

These are *double restriction (bi)categories* (whose vertical morphisms are total) in the sense that we can assign to each module M some $\overline{M}$ which behaves as the restriction idempotent of M.

Question. Should these be inverse categories with joins, with restriction (bi)modules and restriction functors, there will then be a double category of Ehresmann sites, with ideally flat/covering morphisms and inductive functors. How is this related to the category of étendues, geometric morphisms and functors?

Supported Range Modules

Analogous to range categories [7], a range module is a restriction bimodule φ : $\mathbf{X} -\!\!\!\mapsto \mathbf{Y}$ which encodes range in addition to domain: there is an assignment of each $\alpha \in \varphi(y, x)$, to some $\widehat{\alpha} \in \mathbf{Y}(y, y)$ satisfying:

(i) $\overline{\widehat{\alpha}} = \widehat{\alpha}$

(ii) $\widehat{\alpha} \cdot \alpha = \alpha$

(iii) $\widehat{\overline{g} \cdot \alpha} = \overline{g}\widehat{\alpha}$ and $\widehat{\overline{\alpha} \circ f} = \overline{\alpha}\widehat{f}$

(iv) $\widehat{g\widehat{\alpha}} = \widehat{g \cdot \alpha}$ and $\widehat{\alpha \cdot \widehat{f}} = \widehat{\alpha \cdot f}$

If a range module satisfies the stronger condition of $g\widehat{\alpha} = \widehat{g \cdot \alpha} g$ and $\alpha \cdot \widehat{f} = \widehat{\alpha \cdot f} \cdot \alpha$ in place of Condition (iv), we call it a *supported* range module.

We note that an inverse category is always a supported range category with $\overline{f} = f^\circ f$ and $\widehat{f} = f f^\circ$:

$$\widehat{gf}g = gf(gf)^\circ g = gff^\circ g^\circ g = gg^\circ gff^\circ = g\widehat{f}$$

We also note that in an inverse category, we have

$$\overline{f} = f^\circ f = f^\circ f^{\circ\circ} = \widehat{f^\circ}$$

(and similarly, $\widehat{f} = \overline{f^\circ}$). This corresponds to the intuition that a partially invertible morphism is defined exactly by the range of its inverse.

Range modules between categories need to be supported to define composition because the tensor product of two range categories is not a range category unless they are both supported. In this case, the resulting tensor product is also supported, permitting the definition of a bicategory srModule($\mathbf{X}$), where $\mathbf{X}$ is a category, of supported range modules between categories.

Suppose that $\varphi : \mathbf{X} \longmapsto \mathbf{Y}$ is a range module between inverse categories (i.e., an arrow in srModule(**iCat**)). Recall that the equivalence between inverse categories and top-heavy locally inductive groupoids relies on having the restriction and range idempotents (which form the meet-semilattices). It would therefore be interesting to know if the construction $\mathcal{G} : \mathbf{iCat} \to \mathbf{tliGrpd}$ can be extended to a construction

$$\mathcal{G} : \text{srModule}(\mathbf{iCat}) \to \text{srModule}(\mathbf{tliGrpd})$$

and whether this extension is also an equivalence. To start, we construct a supported range module $\mathcal{G}(\varphi)$ between the groupoids $\mathcal{G}(\mathbf{X})$ and $\mathcal{G}(\mathbf{Y})$ corresponding to $\mathbf{X}$ and $\mathbf{Y}$.

Construction 5.4.2. Given a supported range module $\varphi : \mathbf{X} \longrightarrow\!\!\!\!\!\mapsto \mathbf{Y}$, we define a module $\mathcal{G}(\varphi)$ between the corresponding groupoids $\mathcal{G}(\mathbf{X})$ and $\mathcal{G}(\mathbf{Y})$ by

$$\mathcal{G}(\varphi)(\overline{g}, \overline{f}) = \{\alpha \in \varphi(sg, sf) : \overline{\alpha} = \overline{f},\, \widehat{\alpha} = \overline{g}\}$$

We define the left $\mathcal{G}(\mathbf{Y})$-action by

$$(\overline{f} \xrightarrow{\alpha} \overline{g} \xrightarrow{g} \overline{g^\circ}) \longmapsto g \cdot \alpha$$

which is well defined since

$$\overline{g \cdot \alpha} = \overline{\overline{g} \cdot \alpha} = \overline{\widehat{\alpha} \cdot \alpha} = \overline{\alpha} = \overline{f}$$

$$\widehat{g \cdot \alpha} = \widehat{g\widehat{\alpha}} = \widehat{g\overline{g}} = \widehat{g} = \overline{g^\circ}$$

and we define the right $\mathcal{G}(\mathbf{X})$-action by

$$(\overline{f^\circ} \xrightarrow{f^\circ} \overline{f} \xrightarrow{\alpha} \overline{g}) \longmapsto \alpha \cdot f^\circ$$

which is well defined since

$$\overline{\alpha \cdot f^\circ} = \overline{\overline{\alpha} \cdot f^\circ} = \overline{\overline{f} \cdot f^\circ} = \overline{f^\circ}$$

$$\widehat{\alpha \cdot f^{\circ}} = \widehat{\alpha \cdot \widehat{f^{\circ}}} = \widehat{\alpha \cdot \overline{f}} = \widehat{\alpha \cdot \overline{\alpha}} = \widehat{\alpha} = \overline{g} \qquad \diamondsuit$$

Question. Is $\mathcal{G}$ pseudofunctorial? Is it a biequivalence between the module bicategories? This can possibly be proven by extending the functor $\mathcal{I} : \mathbf{tliGrpd} \to \mathbf{iCat}$ to a functor $\mathcal{I} : \text{srModule}(\mathbf{tliGrpd}) \to \text{srModule}(\mathbf{iCat})$ in such a way that $\varphi \cong \mathcal{I}\mathcal{G}(\varphi)$, making $\mathcal{G}$ essentially surjective.

A potential extension of $\mathcal{I}$ can be given by the following construction.

Construction 5.4.3. Given a module $\varphi : \mathbf{G} \longrightarrow\!\!\!\!\!\!\!\mapsto \mathbf{H}$ of top-heavy locally inductive groupoids, we define a module $\mathcal{I}(\varphi)$ between the corresponding inverse categories $\mathcal{I}(\mathbf{G})$ and $\mathcal{I}(\mathbf{H})$ by

$$\mathcal{I}(\varphi)(N, M) = \{\alpha : x \dashrightarrow\!\!\!\!\mapsto y : x \in M, y \in N\} \qquad \diamondsuit$$

Note that

$$\begin{aligned} \mathcal{I}\mathcal{G}(\varphi)(E_y, E_x) &= \{\alpha : \overline{f} \dashrightarrow\!\!\!\!\mapsto \overline{g} \in \mathcal{G}(\varphi) : \overline{f} \in E_x,\, \overline{g} \in E_y\} \\ &= \{\alpha : x \dashrightarrow\!\!\!\!\mapsto y \in \varphi(y, x) : \overline{\alpha} = \overline{f},\, \widehat{\alpha} = \overline{g}\} \end{aligned}$$

We can then define a map $\varphi(y, x) \to \mathcal{I}\mathcal{G}(E_y, E_x)$ by $\alpha : x \dashrightarrow\!\!\!\!\mapsto y \mapsto \alpha : 1_x \dashrightarrow\!\!\!\!\mapsto 1_y$.

Consider the double category $s\mathbb{R}\text{Module}(\mathbf{iCat})$, with supported range modules between inverse categories as horizontal arrows, with functors as vertical arrows and with restriction bimodule morphisms as double cells. Similarly, we define the double category $s\mathbb{R}\text{Module}(\mathbf{tliGrpd})$. We then conjecture the existence of an extension of the functor $\mathcal{I} : \mathbf{tliGrpd} \to \mathbf{iCat}$ to a pseudofunctor $\mathcal{I} : \text{srModule}(\mathbf{tliGrpd}) \to$

srModule(**iCat**) in such a way that $\varphi \cong \mathcal{I}\mathcal{G}(\varphi)$, making $\mathcal{G}$ essentially surjective (cf. Construction 5.4.3).

Conjecture 5.4.4. $s\mathbb{R}$Module(**iCat**) and $s\mathbb{R}$Module(**tliGrpd**) are equivalent as double categories; that is, we extend the Ehresmann-Schein-Nambooripad theorem to two dimensions in an additional way.

Chapter 6: Conclusion

This thesis took advantage of Cockett and Lack's restriction categories, an algebraic abstraction of partial functions, to introduce several categorical structures which form a framework for studying partial computation, certain geometric topological structures and double categories equipped with two compatible restriction structures.

We introduced restriction monads, showed that restriction monads in the bicategory of spans of sets are small restriction categories and showed that the algebras for these monads are certain restriction modules.

We then introduced top-heavy locally inductive groupoids, showed them to be equivalent to inverse categories and equipped them with joins. We showed that the top-heavy locally inductive groupoids corresponding to join inverse categories are Ehresmann sites and we then defined a suitable morphism of Ehresmann sites to be that of an ideally covering and ideally flat functor.

Finally, we studied two dimensional restriction categories, first by defining double restriction categories and second by defining restriction bicategories. We were then able to organize the other data structures of this thesis into a single two-dimensional structure; e.g., restriction modules between restriction monads can be placed in a natural double category and the composition of restriction bimodules gives a natural restriction bicategory.

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